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Supersingular curves on Picard modular surfaces modulo an inert prime

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ABSTRACT

We study the supersingular curves on Picard modular surfaces modulo a prime p which is inert in the underlying quadratic imaginary field. We analyze the automorphic vector bundles in characteristic p, and as an application derive a formula relating the number of irreducible components in the supersingular locus to the second Chern class of the surface.

The characteristic p fibers of Shimura varieties of PEL type admit special subvarieties over which the abelian varieties which they parametrize become supersingular. These special subvarieties and the automorphic vector bundles over them have become the subject of extensive research in recent years. The purpose of this note is to analyze one of the simplest examples beyond Shimura curves, namely Picard modular surfaces, at a prime p which is inert in the underlying quadratic imaginary field. The supersingular locus on these surfaces has been studied in depth by Bültel, Vollaard and Wedhorn

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([6,31,32]) and our debt to these authors should be obvious. It consists of a collection of Fermat curves of degree $p + 1$, intersecting transversally at the *superspecial* points. We complement their work by analyzing the automorphic vector bundles in characteristic p, in relation to the supersingular strata. Although some of our results (e.g. the construction of a Hasse invariant, see [12]) have been recently generalized to much larger classes of Shimura varieties, focusing on this simple example allows us to give more details. As an example, we derive the following theorem (3.3).

Theorem 0.1. *Let $\mathcal{K}$ be a quadratic imaginary field and let $\bar{S}$ be the Picard modular surface of level $N \geq 3$ associated with $\mathcal{K}$ (for a precise definition see the text below). Let p be a prime which is inert in $\mathcal{K}$ and relatively prime to $2N$. Then the number of irreducible components in the supersingular locus of $\bar{S}$ $\mathrm{mod}\ p$ is $c_2(\bar{S})/3$, where $c_2(\bar{S})$ is the second Chern class of $\bar{S}$.*

Thanks to results of Holzapfel, this number is expressed in terms of the value of the L-function $L(s, (D_{\mathcal{K}}/\cdot))$ at $s = 3$, see Theorem 1.2.

Although the uniformization of the supersingular locus by Rapoport–Zink spaces yields a group-theoretic parametrization of the irreducible components by a certain double coset space, this parametrization in itself is not sufficient to yield the theorem. Our proof goes along different lines, invoking intersection theory on $\bar{S}$.

Note that the number of irreducible components comes out to be independent of p. A similar result was obtained in [2] for Hilbert modular surfaces. This is different from the situation with the Siegel modular variety A_g. We are still lacking a conceptual understanding of this independence of p.

The first section introduces notation and background. The second contains most of the analysis modulo p. The Picard surface modulo p has 3 strata: the μ-ordinary (generic) stratum $\bar{S}_\mu$, the general supersingular locus S_{gss} and the superspecial points S_{ssp}. There are two basic automorphic vector bundles to consider, a rank 2 bundle $\mathcal{P}$ and a line bundle $\mathcal{L}$. Of particular importance are the Verschiebung homomorphisms $V_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}^{(p)}$ and $V_{\mathcal{L}} : \mathcal{L} \to \mathcal{P}^{(p)}$. It turns out that outside the superspecial points $V_{\mathcal{P}}$ and $V_{\mathcal{L}}$ are both of rank 1, but that the supersingular locus is *defined* by the equation $V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{L}} = 0$. The Hasse invariant $h_{\bar{\Sigma}} = V_{\mathcal{P}}^{(p)} \circ V_{\mathcal{L}}$ is a canonical global section of $\mathcal{L}^{p^2-1}$ whose divisor is $S_{ss} = S_{gss} \cup S_{ssp}$. On S_{ss}, in turn, we construct a canonical section h_{ssp} of $\mathcal{L}^{p^3+1}$, which vanishes precisely at the superspecial points S_{ssp} (to a high order). This secondary Hasse invariant is related to more general work of Boxer [5].

In the last section we use the information carried by h_{ssp}, together with some intersection theoretic computations, to prove the above theorem.

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