Journal of Number Theory 171 (2017) 449–473

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Bounded gaps between Gaussian primes

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A R T I C L E I N F O

Article history:
Received 3 November 2015
Received in revised form 26 July 2016
Accepted 26 July 2016
Available online 6 September 2016
Communicated by R.C. Vaughan

MSC:
11N36
11R44

Keywords:
Primes of the form $a^2 + b^2$
Gaussian primes
Bounded gaps
Higher rank Selberg sieve

A B S T R A C T

We show that there are infinitely many distinct rational primes of the form $p_1 = a^2 + b^2$ and $p_2 = a^2 + (b + h)^2$, with a, b, h integers, such that $|h| \leq 246$. We do this by viewing a Gaussian prime $c + di$ as a lattice point (c, d) in $\mathbb{R}^2$ and showing that there are infinitely many pairs of distinct Gaussian primes (c_1, d_1) and (c_2, d_2) such that the Euclidean distance between them is bounded by 246. Our method, motivated by the work of Maynard [9] and the Polymath project [13], is applicable to the wider setting of imaginary quadratic fields with class number 1 and yields better results than those previously obtained for gaps between primes in the corresponding number rings.

1. Introduction

The well-known twin prime conjecture states that there are infinitely many primes p such that $p + 2$ is also prime. The last decade has marked remarkable progress towards this conjecture. We now know that there are infinitely many bounded gaps between the primes and moreover this bound has been reduced to as low as 246. This progress began with the first such result by Zhang [19], building innovatively upon previous work of

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http://dx.doi.org/10.1016/j.jnt.2016.07.008
0022-314X/

Goldston, Pintz and Yıldırım [3], followed by important contributions due to Maynard [9] and Tao, and eventually culminated in the collaboration of a number of mathematicians under the Polymath project [13].

One way to think about the twin prime conjecture is to view it as a special case of the more general prime k-tuples conjecture. A set $\mathscr{H}$ of distinct non-negative integers is said to be admissible in $\mathbb{Z}$ if for every prime p, there is some residue class which is not contained in $\mathscr{H} \pmod p$, that is,

$$|\mathscr{H} \pmod p| < p \text{ for all primes } p.$$

Then, the prime k-tuples conjecture asserts that for any given admissible set $\mathscr{H}$, there are infinitely many n such that $n + h_1, \ldots, n + h_k$ are all prime. To view the twin prime conjecture as a special case of this, notice that any admissible set of size 2 is given without loss of generality by $\{0, 2m\}$, where $m \in \mathbb{Z}^+$. Then, the prime k-tuples conjecture applied to this set with $m = 1$ yields infinitely many twin primes while any $m \geq 2$ gives an infinitude of what are known as generalized twin prime pairs.

We will formulate the analogue of the generalized twin prime conjecture in the setting of the Gaussian integers $\mathbb{Z}[i]$. In order to do this, we first define the notions of primality and admissibility in $\mathbb{Z}[i]$. We say that $\mathbf{p} \in \mathbb{Z}[i]$ is a Gaussian prime if the ideal $(\mathbf{p}) \subseteq \mathbb{Z}[i]$ generated by it is a prime ideal. A set $\mathscr{H} = \{\mathbf{h}_1, \ldots, \mathbf{h}_k\}$ of distinct Gaussian integers is called admissible if $|\mathscr{H} \pmod{\mathbf{p}}| < |N(\mathbf{p})|$ for every Gaussian prime $\mathbf{p}$. Let us consider admissible sets of size 2. It is clear that admissibility must now be checked only for the primes $1 + i$ and $1 - i$ having norm 2. As $m_1 + m_2 i \pmod{1 \pm i} \equiv 0$ iff m_1 and m_2 have the same parity, without loss of generality, any admissible set of size 2 is given by $\{0, m_1 + m_2 i\}$, where $m_1 \equiv m_2 \pmod 2$. This gives us the following analogue of the generalized twin prime conjecture:

Conjecture 1.1 *(Generalized twin primes in $\mathbb{Z}[i]$). Given any integers m_1 and m_2 having the same parity, there are infinitely many Gaussian primes $\mathbf{p} = a + bi$ such that $a + m_1 + (b + m_2)i$ is also prime.*

This conjecture has an interesting consequence in terms of the rational primes obtained by taking norms of the Gaussian primes above.

Conjecture 1.2 *(Pairs of rational primes of a special form). Given any integers m_1 and m_2 having the same parity, there are infinitely many pairs of rational primes (p_1, p_2) of the form $p_1 = a^2 + b^2$ and $p_2 = (a + m_1)^2 + (b + m_2)^2$.*

It is indeed intriguing that the problem of finding infinitely many *pairs* of primes of a special form in $\mathbb{Z}$ translates to the generalized twin prime conjecture when viewed in the ring $\mathbb{Z}[i]$. This problem is still unsolved. A related recent result by Thorner [17] states that for any fixed $0 < \epsilon < 1/2$, there are infinitely primes p_1, p_2 of the form $p = a^2 + b^2$ with $|a| < \epsilon\sqrt{p}$, such that $|p_1 - p_2| \leq C(\epsilon)$.