Journal of Number Theory 171 (2017) 495–526

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Almost all hyperharmonic numbers are not integers

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A R T I C L E I N F O

Article history:
Received 25 April 2016
Accepted 13 July 2016
Available online 6 September 2016
Communicated by D. Goss

MSC:
11B83
5A10
11B75

Keywords:
Hyperharmonic numbers
Harmonic numbers
Prime number theory

A B S T R A C T

It is an open question asked by Mezö that there is no hyperharmonic integer except 1. So far it has been proved that all hyperharmonic numbers are not integers up to order $r = 25$. In this paper, we extend the current results for large orders. Our method will be based on three different approaches, namely analytic, combinatorial and algebraic. From analytic point of view, by exploiting primes in short intervals we prove that almost all hyperharmonic numbers are not integers. Then using combinatorial techniques, we show that if n is even or a prime power, or r is odd then the corresponding hyperharmonic number is not integer. Finally as algebraic methods, we relate the integerness property of hyperharmonic numbers with solutions of some polynomials in finite fields.

1. Introduction

The goal of this paper is to analyse the integerness property of hyperharmonic numbers. We answer Mezö's question [18] in almost all cases, which states that all hy-

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http://dx.doi.org/10.1016/j.jnt.2016.07.023
0022-314X/

perharmonic numbers are not integers except 1. In the same paper, he also proved that there are none of them of order $r = 2, 3$. In [5,6], this was extended to $r \leq 25$ and they gave a set of integer pairs (n, r) such that $h_n^{(r)}$ is not an integer. In the current paper, we extend all these results in the literature applying analytic, combinatorial and algebraic tools.

Harmonic numbers are defined as the sequence of partial sums of the harmonic series

$$h_n = \sum_{k=1}^{n} \frac{1}{k}$$

for $n \geq 1$. These numbers have been studied extensively and they are equipped with a lot of combinatorial and analytic properties. As a combinatorial one, it was proved that there is no harmonic number which is an integer except 1 [21]. This can also be seen by Bertrand's postulate, which was reproved by Erdös [14] from combinatorial point of view. Although almost all harmonic numbers are not integers, Wolstenholme [22] proved that $h_{p-1} \equiv 0 \bmod p^2$ for all primes $p \geq 5$. More recently this result was extended by Alkan in [1]. On the analytical side, Euler's harmonic zeta function which is defined by

$$\zeta_h(s) = \sum_{n=1}^{\infty} \frac{h_n}{n^s}$$

for $\Re(s) > 1$ satisfies the well-known striking relation (see [10, p. 252])

$$2\zeta_h(m) = (m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1) \tag{1.1}$$

for all $m \geq 2$, where the sum is zero when $m = 2$ and $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $\Re(s) > 1$. Euler's formula yields some important evaluations

$$\sum_{n=1}^{\infty} \frac{h_n}{n^2} = 2\zeta(3), \quad \sum_{n=1}^{\infty} \frac{h_n}{n^3} = \frac{5}{4}\zeta(4) = \frac{\pi^4}{72}.$$

Recently, by making use of connections to log-sine integrals which have remarkable applications to physical problems via potential energy of charged particle systems on the unit circle, Alkan (see [2] and [3]) showed that real numbers can be strongly approximated by combinations of values of $\zeta_h(s)$ and $\zeta(s)$, which resembles the classical result of Liouville's theorem on Diophantine approximation.