Journal of Number Theory 162 (2016) 1–10

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the Mixed Littlewood Conjecture and continued fractions in quadratic fields

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ARTICLE INFO

Article history:
Received 6 April 2015
Accepted 12 October 2015
Available online 10 December 2015
Communicated by David Goss

MSC:
11J13
11J61
11J68

Keywords:
Littlewood Conjecture
Simultaneous Diophantine approximation

ABSTRACT

We show how a recent result by Aka and Shapira on the evolution of continued fractions in a fixed quadratic field implies a strengthened form of the classic result of de Mathan and Teulié on the Mixed Littlewood Conjecture.

1. Introduction

A famous open problem in the field of Diophantine approximation is the Littlewood Conjecture which claims that, for every pair (α, β) of real numbers, we have

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| \ = \ 0, \tag{1}$$

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http://dx.doi.org/10.1016/j.jnt.2015.10.007
0022-314X/

where $\|\cdot\|$ denotes the distance to the nearest integer. The first significant contribution to this question is due to Cassels and Swinnerton-Dyer [5] who proved that (1) is satisfied when α and β belong to the same cubic field. This result was sharpened by Peck [11] who showed that if 1, α, β form a basis of a cubic field, then

$$\liminf_{q\to\infty} q \cdot \log q \cdot \|q\alpha\| \cdot \|q\beta\| \;<\; \infty.$$

Further examples of pairs (α, β) satisfying (1) are given in [1] and [6]. Despite some recent progress on the Hausdorff dimension of the set of counterexamples [12,8], the Littlewood Conjecture remains an open problem.

In 2004, de Mathan and Teulié [7] proposed a variant of the Littlewood Conjecture, called Mixed Littlewood Conjecture, in which the quantity $\|q\beta\|$ is replaced by a pseudo-absolute value $|q|_{\mathcal{D}}$. A pseudo-absolute sequence $\mathcal{D}$ is an increasing sequence of positive integers $\mathcal{D} = (u_n)_{n\in\mathbb{N}}$ with $u_1 = 1$ and $u_n | u_{n+1}$ for all n. The pseudo-absolute value $|q|_{\mathcal{D}}$ is then defined by

$$|q|_{\mathcal{D}} = \inf \{1/u_n \;:\; q \in u_n\mathbb{Z}\}.$$

When $\mathcal{D}$ is the sequence $(p^n)_{n\in\mathbb{N}}$, where p is a prime number, then $|\cdot|_{\mathcal{D}}$ is the usual p-adic value $|\cdot|_p$, normalised such that $|p|_p = p^{-1}$, and de Mathan and Teulié's conjecture is then known as p-adic Littlewood Conjecture.

Mixed Littlewood Conjecture. *For every real number α and every pseudo-absolute sequence $\mathcal{D}$, we have*

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} \;=\; 0. \tag{2}$$

This conjecture obviously holds when α is rational or has unbounded partial quotients. Thus one only has to consider the case when α is a badly approximable number (that is irrational α with bounded partial quotients). From a metric point of view, the set of badly approximable numbers is moderately small: it has Lebesgue measure 0 but Hausdorff dimension 1.

In 2007, Einsiedler and Kleinbock [9] established that, for every given prime number p, the set of counterexamples α to the p-adic Littlewood Conjecture has Hausdorff dimension 0, so this set is much smaller than the set of all badly approximable numbers. In an opposite direction, Badziahin and Velani [3] proved that, for every given pseudo-absolute sequence $\mathcal{D}$, the set of real numbers α satisfying

$$\inf_{q\leq 3} q \cdot \log q \cdot \log\log q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} \;>\; 0 \tag{3}$$

has full Hausdorff dimension.

However, the first contribution to de Mathan and Teulié's conjecture goes back to themselves in [7]. A natural family of badly approximable numbers are the quadratic