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An arithmetical approach to the convergence problem of series of dilated functions and its connection with the Riemann Zeta function

Michel J.G. Weber

IRMA, 10 rue du Général Zimmer, 67084 Strasbourg Cedex, France

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ABSTRACT

Given a periodic function f, we study the convergence almost everywhere and in norm of the series $\sum_k c_k f(kx)$. Let $f(x) = \sum_{m=1}^{\infty} a_m \sin 2\pi mx$ where $\sum_{m=1}^{\infty} a_m^2 d(m) < \infty$ and $d(m) = \sum_{d|m} 1$, and let $f_n(x) = f(nx)$. We show by using a new decomposition of squared sums that for any $K \subset \mathbb{N}$ finite, $\| \sum_{k\in K} c_k f_k \|_2^2 \le (\sum_{m=1}^{\infty} a_m^2 d(m)) \sum_{k\in K} c_k^2 d(k^2)$. If $f^{(s)}(x) = \sum_{j=1}^{\infty} \frac{\sin 2\pi jx}{j^s}$, $s > 1/2$, by only using elementary Dirichlet convolution calculus, we show that for $0 < \varepsilon \le 2s-1$, $\zeta(2s)^{-1} \| \sum_{k\in K} c_k f_k^{(s)} \|_2^2 \le \frac{1+\varepsilon}{\varepsilon} (\sum_{k\in K} |c_k|^2 \sigma_{1+\varepsilon-2s}(k))$, where $\sigma_h(n) = \sum_{d|n} d^h$. From this we deduce that if $f \in \mathrm{BV}(\mathbb{T})$, $\langle f, 1\rangle = 0$ and

$$\sum_k c_k^2 \frac{(\log\log k)^4}{(\log\log\log k)^2} < \infty,$$

then the series $\sum_k c_k f_k$ converges almost everywhere. This slightly improves a recent result, depending on a fine analysis on the polydisc [1, th. 3] ($n_k = k$), where it was assumed that $\sum_k c_k^2 (\log\log k)^\gamma$ converges for some $\gamma > 4$. We further show that the same conclusion holds under the arithmetical condition

$$\sum_k c_k^2 (\log\log k)^{2+b} \sigma_{-1+\frac{1}{(\log\log k)^{b/3}}}(k) < \infty,$$

for some $b > 0$, or if $\sum_k c_k^2 d(k^2)(\log\log k)^2 < \infty$. We also derive from a recent result of Hilberdink an Ω-result for the

E-mail address: michel.weber@math.unistra.fr.

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Riemann Zeta function involving factor closed sets. As an application we find that simple conditions on T and ν ensuring that for any $\sigma > 1/2$, $0 \le \varepsilon < \sigma$, we have

$$\max_{1\le t\le T} |\zeta(\sigma+it)| \ge C(\sigma)\Big(\frac{1}{\sigma_{-2\varepsilon}(\nu)}\sum_{n|\nu}\frac{\sigma_{-s+\varepsilon}(n)^2}{n^{2\varepsilon}}\Big)^{1/2}.$$

We finally prove an important complementary result to Wintner's famous characterization of mean convergence of series $\sum_{k=0}^{\infty} c_k f_k$.

1. Introduction

Given a periodic function f and an increasing sequence $\mathcal{N} = \{n_k, k \ge 1\}$ of positive integers, one can formally define the series $\sum_{k=1}^{\infty} c_k f(n_k x)$ and ask under which conditions this series converges in norm or almost everywhere, for instance for any real coefficient sequence $\underline{c} = (c_k)_k \in \ell^2(\mathbb{N})$. This is one of the oldest and most central problems in the theory of systems of dilated sums. We only briefly outline the kind of results obtained. First studies were made at the beginning of the last century (see Jerosch and Weyl [24] where a.e. convergence is obtained under growth conditions on coefficients and Fourier coefficients of f), parallel to similar ones for the trigonometrical system. This partly explains why until Carleson's famous proof of Lusin's hypothesis, the results obtained essentially concerned functions with slowly growing modulus or integral modulus of continuity and/or sequences $\mathcal{N}$ verifying the classical Hadamard gap condition: $n_{k+1}/n_k \ge q > 1$ for all k. Carleson's result triggered a new interest, permitting substantial progresses in this direction, under the main impulse of Russian analysts, among them Gaposhkin and later by Berkes. We refer to [4] for more details and references. Then the attention to these problems declined until very recently where there is a renewed activity, notably concerning their connection with some questions (Ω-results) on the Riemann Zeta function.

In analogy with parallel questions concerning partial sums $\sum_{k=1}^{n} f(kx)$, $n = 1, 2, \dots$, strong law of large numbers, studied by Gál, Koksma (see also [6]), and law of the iterated logarithm, central limit theorem, invariance principle, much explored by Erdös, Berkes and Philipp, and Gaposhkin notably, recent works show that the arithmetical nature of the support of the coefficient sequence, as well as the analytic nature of f, interact in a complex way in the study of the convergence almost everywhere and in norm of these series. The part of the theory devoted to individual results, namely the search of convergence conditions linking f, $\mathcal{N}$ and $\underline{c}$ is, to say the least, barely investigated. Our main concern in this work is precisely the search of individual conditions ensuring the almost everywhere convergence of the series $\sum_{k=1}^{\infty} c_k f(n_k x)$. We propose new approaches for treating these questions. Notice before continuing, that the problem under consid-