Journal of Number Theory 162 (2016) 180–189

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On monochromatic sums of squares of primes

Guohua Chen

Department of Mathematics, Shandong University, Jinan, Shandong, 250100, China

A R T I C L E   I N F O

Article history:
Received 2 August 2015
Received in revised form 13 October 2015
Accepted 13 October 2015
Available online 14 December 2015
Communicated by David Goss

MSC:
11P32
11P55

Keywords:
Monochromatic sums of squares of primes
Circle method
Exponential sums over primes
Large sieve for quadratic amplitudes

A B S T R A C T

Let K be a positive integer, $\{A_i,\ 1 \le i \le K\}$ be any partition of the sequence of squares of primes and $s(K)$ be the smallest positive integer such that every sufficiently large integer can be written as the sum of no more than $s(K)$ elements, which belong to one of the sets A_i. In this paper, we prove that $s(K) \ll_\epsilon K^{2+\epsilon}$ for sufficiently small positive number ϵ and all $K \ge 1$.

1. Introduction

A consequence of three primes theorem asserts that every integer larger than 1 can be expressed as a sum of no more than four prime numbers. A classical theorem of Lagrange asserts that every positive integer can be written as a sum of four squares of natural numbers. Sárközy [8] asked for a chromatic version of these theorems. Hua's theorem with five squares of primes is similar to the previous two theorems, which asserts that

E-mail address: guohuachen@mail.sdu.edu.cn.

http://dx.doi.org/10.1016/j.jnt.2015.10.017
0022-314X/

every sufficiently large integer congruent to 5 (mod 24) can be expressed as a sum of five squares of primes, then we can conclude that every sufficiently large integer can be expressed as a sum of no more than a fixed number of squares of primes. In this paper, we consider a chromatic version of this conclusion. More precisely, let A denote the sequence of squares of primes and suppose $K \geq 1$ is an integer. Then the problem is to determine the upper bounds in terms of K for the smallest integer $s(K)$ with the property that there exists an integer $n(K)$ such that given any partition $\{A_i, 1 \leq i \leq K\}$ of the sequence of the squares of primes, there is an integer i with $1 \leq i \leq K$ for each integer $n \geq n(K)$ so that n can be written as the sum of no more than $s(K)$ squares of primes, which belong to one of the sets A_i.

Hegyvári and Hennecart [2] used an essentially elementary method to attack the problem of Sárközy for both the sequence of squares and the sequence of prime numbers. Applying their method to the sequence of squares, they obtained the bound $s(K) \ll (K\log K)^5$. Applying their method to the sequence of prime numbers, they obtained the bound $s(K) \leq 1500K^3$. Ramana and Ramaré [6] obtained $s(K) \leq 1700CK\log\log K$ improving on the upper bound $s(K) \ll K^3$ for the sequence of the prime numbers. Akhilesh and Ramana [1] obtained $s(K) \ll_\epsilon K^{2+\epsilon}$ for the sequence of squares, where ϵ was any positive real number, improving on the upper bound given by Hegyvári and Hennecart.

Motivated by [1], this paper considers the upper bound of $s(K)$ for the sequence of squares of primes and obtains the following main theorems.

Theorem 1.1. *For any integer $K \geq 1$ and sufficiently small positive number ϵ, we have $s(K) \ll_\epsilon K^{2+\epsilon}$.*

The proof of Theorem 1.1 needs the following theorem, which we state with the aid of the following notation. For any subset $\mathcal{A}$ of the squares of primes in the interval $(N, 4N]$, we shall write

$$E_6(\mathcal{A}) = \sum_{q_1^2+p_1^2+\ldots+p_5^2=q_2^2+p_6^2+\ldots+p_{10}^2} (\log q_1)(\log q_2)(\log p_1)\cdots(\log p_{10}),$$

where q_1^2, q_2^2, $p_i^2 \in \mathcal{A}$ with $1 \leq i \leq 10$.

Theorem 1.2. *Let $K \geq 1$ be a positive integer and ϵ be a sufficiently small positive number. Then there exists an integer N_0, depending only on K and ϵ, such that for all $N \geq N_0$ and any subset $\mathcal{A}$ of the squares of primes in the interval $(N, 4N]$ with $|\mathcal{A}| \gg N^{\frac{1}{2}}/K\log N$, we have*

$$E_6(\mathcal{A}) \ll_\epsilon |\mathcal{A}|^{10} L^{10} K^\epsilon.$$

Notation. Throughout, the letter ϵ will denote sufficiently small positive number, the (a, b, c, d) and $[a, b, c, d]$ will denote a tuple and the least common multiple respectively, p_i and q_i will denote prime numbers, L will denote the $\log N$ and $e(z)$ will denote $e^{2\pi iz}$.