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Inequalities and asymptotic expansions for the gamma function related to Mortici's formula

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A B S T R A C T

Inspired by a formula of C. Mortici for approximation to $\Gamma(n+1)$, we establish a class of asymptotic expansions for the gamma function. Based on these expansions, we present new upper and lower bounds for the gamma function.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1.1}$$

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has many applications in statistical physics, probability theory and number theory. Actually it was discovered by A. De Moivre (1667–1754) in the form

$$n! \sim C \cdot \sqrt{n}(n/e)^n,$$

and Stirling (1692–1770) identified the constant C precisely $\sqrt{2\pi}$. Stirling's formula has attracted much interest of many mathematicians and has motivated a large number of research papers concerning various generalizations and improvements (see [4–14,16, 18–35,37,38,41] and the references cited therein).

The following asymptotic formulas are well-known for the gamma function (see, for example, [1, p. 257]):

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right) \\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \cdots\right) \\ &\qquad (x \to \infty) \quad \text{(Stirling series)}\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51\,840x^3} - \frac{571}{2\,488\,320x^4} + \cdots\right) \\ &\qquad (x \to \infty) \quad \text{(Laplace formula)},\end{aligned} \tag{1.3}$$

where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), $\mathbb{N} := \{1, 2, \ldots\}$) are the nth Bernoulli numbers defined by the following generating function (see, for example, [39, Section 1.7]):

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi).$$

The Laplace formula (1.3) is sometimes incorrectly called Stirling series (see [15, pp. 2–3]).

Stirling's formula is in fact the first approximation to the asymptotic formula (1.3). Inspired by (1.1), Burnside [8] found a slightly more accurate approximation than Stirling's formula as follows:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}. \tag{1.4}$$

The formula (1.4) was rediscovered by Weissman [41], Spouge [38] and Schuster [37]. By replacing in the Stirling's formula $\sqrt{2n}$ by $\left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6}$, Ramanujan [36, p. 339] presented the following approximation: