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Patterns in numbers and infinite sums and products

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A B S T R A C T

Text. Let $a_{w,B}(n)$ denote the number of occurrences of the word w in the base B expansion of the non-negative integer n. In this article we generalize the results of Allouche and Shallit [2] by proving the existence of a finite set $L_{w,B}$ of pairs (l, c_l) where l is a polynomial with integer coefficients of degree 1 and c_l an integer such that:

$$\sum_{n\geq 0}(-1)^{a_{w,B}(n)}\sum_{(l,c_l)\in L_{w,B}} c_l f(l(n)) = \begin{cases} 0 & \text{if } w \neq 0^j, \\ -2\cdot(-1)^{a_{w,B}(0)} f(0) & \text{if } w = 0^j \end{cases}$$

where f is any function that verifies certain convergence conditions.
After exponentiating, we recover previous results and obtain new ones such as

$$\prod_{n\geq 1}\left(\frac{3n+1}{3n+2}\right)^{(-1)^n} = \frac{2}{\sqrt{3}},$$

and

$$\prod_{n\geq 1}\left(\frac{9n+7}{9n+8}\right)^{(-1)^{a_{21,3}(n)}} = \frac{8}{7\sqrt{3}}.$$

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Video. For a video summary of this paper, please visit https://youtu.be/0wXL7xPkooc.

1. Introduction

Let $s_q(n)$ denote the sum of digits of the non-negative integer n when written in base q. Woods and Robbins [6,7] proved that

$$\prod_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{s_2(n)}} = \frac{\sqrt{2}}{2}. \tag{1}$$

Allouche and Shallit [2] looked at the function $a_w(n)$, defined as the number of occurrences of the finite non-empty binary word w in the binary expansion of n. With this notation the $s_2(n)$ in Equation (1) becomes $(-1)^{a_1(n)}$. With the following two theorems, they generalized the result to $a_w(n)$ for all w.

Theorem 1. *(See Allouche and Shallit [2].) Let w be a string of zeros and ones, and*

$$g = 2^{|w|-1}, \quad h = \lfloor v(w)/2 \rfloor,$$

and let X be a complex number with $|X| \leq 1$ and $X \neq 1$. Then

$$\sum_n X^{a_w(gn+h)} L(2gn + v(w)) = -\frac{1}{1-X},$$

where the sum is over $n \geq 1$ for $w = 0^j$ and $n \geq 0$ otherwise.

Theorem 2. *(See Allouche and Shallit [2].) There is an effectively computable rational function $b_w(n)$ such that, for all $X \neq 1$ with $|X| \leq 1$, we have*

$$\sum_n \log_2(b_w(n)) X^{a_w(n)} = -\frac{1}{1-X}, \tag{2}$$

where the sum is over $n \geq 1$ for $w = 0^j$ and $n \geq 0$ otherwise.

By setting $X = -1$, $w = 1$ in equation (2) and exponentiating we rediscover equation (1). Other values of w give new results; for example,

$$\prod_{n\geq 0}\left(\frac{(4n+2)(8n+7)(8n+3)(16n+10)}{(4n+3)(8n+6)(8n+2)(16n+11)}\right)^{(-1)^{a_{1010}(n)}} = \frac{\sqrt{2}}{2}. \tag{3}$$