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Summation identities and special values of hypergeometric series in the p-adic setting ☆

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We prove hypergeometric type summation identities for a function defined in terms of quotients of the p-adic gamma function by counting points on certain families of hyperelliptic curves over $\mathbb{F}_q$. We also find certain special values of that function.

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1. Introduction and statement of results

In [11], Greene introduced the notion of hypergeometric functions over finite fields analogous to classical hypergeometric series. Since then many interesting relations between special values of these hypergeometric functions and the number of points on certain varieties over finite fields have been obtained. The arguments of these functions are multiplicative characters of finite fields and, consequently, results involving hypergeometric functions over finite fields are often restricted to primes in certain congruence classes. For example, the expressions for the trace of Frobenius map on families of elliptic curves given in [2,3,9,15,16] are restricted to certain congruence classes to facilitate the existence of characters of specific orders. To overcome these restrictions, in [18,19], the third author defined a function ${}_nG_n[\cdots]$ in terms of quotients of the p-adic gamma function which can best be described as an analogue of hypergeometric series in the p-adic setting. He showed [17–19] how results involving hypergeometric functions over finite fields can be extended to almost all primes using the function ${}_nG_n[\cdots]$.

Let p be an odd prime, and let $\mathbb{F}_q$ denote the finite field with q elements. Let $\Gamma_p(\cdot)$ denote the Morita's p-adic gamma function, and let ω denote the Teichmüller character of $\mathbb{F}_q$ with $\bar{\omega}$ denoting its character inverse. For $x \in \mathbb{Q}$ we let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and $\langle x \rangle$ denote the fractional part of x.

Definition 1.1. (See [19, Definition 5.1].) Let $q = p^r$, for p an odd prime and $r \in \mathbb{Z}^+$, and let $t \in \mathbb{F}_q$. For $n \in \mathbb{Z}^+$ and $1 \leq i \leq n$, let $a_i, b_i \in \mathbb{Q} \cap \mathbb{Z}_p$. Then we define

$$
{}_nG_n\left[\begin{matrix} a_1, & a_2, & \ldots, & a_n \\ b_1, & b_2, & \ldots, & b_n \end{matrix} \,|t\right]_q := \frac{-1}{q-1}\sum_{j=0}^{q-2}(-1)^{jn} \quad \bar{\omega}^j(t)
$$

$$
\times \prod_{i=1}^{n}\prod_{k=0}^{r-1}(-p)^{-\lfloor\langle a_i p^k\rangle - \frac{jp^k}{q-1}\rfloor - \lfloor\langle -b_i p^k\rangle + \frac{jp^k}{q-1}\rfloor} \frac{\Gamma_p(\langle(a_i - \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle a_i p^k\rangle)} \frac{\Gamma_p(\langle(-b_i + \frac{j}{q-1})p^k\rangle)}{\Gamma_p(\langle -b_i p^k\rangle)}.
$$

We note that the value of ${}_nG_n[\cdots]$ depends only on the fractional part of the parameters a_i and b_i, and is invariant if we change the order of the parameters.

The aim of this paper is to prove summation identities for ${}_nG_n[\cdots]$. In [19], the third author showed that transformations for hypergeometric functions over finite fields can be re-written in terms of ${}_nG_n[\cdots]$. However, such transformations will only hold for all p where the original characters existed over $\mathbb{F}_q$, and hence restricted to primes in certain congruence classes. It is a non-trivial exercise to then extend these results to almost all primes. While numerous transformations exist for the finite field hypergeometric functions, very few exist for ${}_nG_n[\cdots]$ in full generality. The first and second authors [5,6] provide transformations for ${}_2G_2[\cdots]_q$ by counting points on various families of elliptic curves over $\mathbb{F}_q$. Recently, the third author and Fuselier [10] provide two transformations for ${}_nG_n[\cdots]_p$ when $n = 3$ and $n = 4$, respectively. They also provide two transformations for ${}_nG_n[\cdots]_p$ for any n. In this paper we prove eight summation identities for the function