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Applications of stuffle product of multiple zeta values

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A B S T R A C T

We obtain some formulas for the stuffle product and apply them to derive a decomposition formula for multiple zeta values. Moreover, we give an application to combinatorics and get the following identity:

$$D(n+1,t+1)+D(n,t) = 2\sum_{\ell=0}^{n} D(t,n-\ell)+2\sum_{\ell=0}^{t} D(n,t-\ell),$$

where $D(n,t)$ is the Delannoy number.

1. Introduction

The multiple zeta value (MZV) is defined by

$$\zeta(\boldsymbol{\alpha}) = \sum_{n_1>n_2>\cdots>n_r>0} n_1^{-\alpha_1} n_2^{-\alpha_2} \cdots n_r^{-\alpha_r},$$

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where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is an r-tuple of positive integers with $\alpha_1 \geq 2$. The number r is called the depth of $\zeta(\boldsymbol{\alpha})$ and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is called the weight of $\zeta(\boldsymbol{\alpha})$. For convenience, we write $\{s\}^k$ to be k repetitions of s, for example, $\zeta(\{s\}^3) = \zeta(s, s, s)$, and in particular $\zeta(t, \{s\}^0) = \zeta(t)$.

Recently multiple zeta values (MZVs) and their generalizations have attracted much attention, both in pure mathematics and theoretical physics (see [3]). A systematic study only started in the early 1990s, although the prehistory can be traced back to Euler in the 18th century.

A principal goal in the theoretical study of MZVs is to determine all possible algebraic relations among them. Several explicit values are interesting and known for special index sets (e.g. [1,2,9,10]). For example, Zagier [10] evaluated the value of $\zeta(\{2\}^a, 3, \{2\}^b)$ and T. Arakawa and M. Kaneko (ref. [9, Theorem 1]) evaluated the value of $\zeta(\{2n\}^m)$.

Let us consider the coding of multi-indices $\vec{s} = (s_1, \ldots, s_k)$, s_i are positive integers and $s_1 > 1$, by words (that is, by monomials in non-commutative variables) over the alphabet $X = \{x, y\}$ by the rule

$$\vec{s} \mapsto x_{\vec{s}} = x^{s_1-1} y x^{s_2-1} y \cdots x^{s_k-1} y.$$

We set

$$\zeta(x_{\vec{s}}) := \zeta(\vec{s})$$

for all admissible words (that is, beginning with x and ending with y); then the weight (or the degree) $|x_{\vec{s}}| := |\vec{s}|$ coincides with the total degree of the monomial $x_{\vec{s}}$, whereas the length (or the depth) $l(x_{\vec{s}}) := l(\vec{s})$ is the degree with respect to the variable y.

Let $\mathbb{Q}\langle X\rangle = \mathbb{Q}\langle x, y\rangle$ be the $\mathbb{Q}$-algebra of polynomials in two non-commutative variables which is graded by the degree (where each of the variables x and y is assumed to be of degree 1); we identify the algebra $\mathbb{Q}\langle X\rangle$ with the graded $\mathbb{Q}$-vector space $\mathfrak{H}$ spanned by the monomials in the variables x and y (see [7]).

We also introduce the graded $\mathbb{Q}$-vector spaces $\mathfrak{H}^1 = \mathbb{Q}\mathbf{1} \bigoplus \mathfrak{H}y$ and $\mathfrak{H}^0 = \mathbb{Q}\mathbf{1} \bigoplus x\mathfrak{H}y$, where $\mathbf{1}$ denotes the unit (the empty word of weight 0 and length 0) of the algebra $\mathbb{Q}\langle X\rangle$. Then the space $\mathfrak{H}^1$ can be regarded as the subalgebra of $\mathbb{Q}\langle X\rangle$ generated by the words $z_s = x^{s-1}y$, whereas $\mathfrak{H}^0$ is the $\mathbb{Q}$-vector space spanned by all admissible words.

Let us define the shuffle product ш on $\mathfrak{H}$ and the stuffle product $*$ (the *harmonic product*) on $\mathfrak{H}^1$ by the rules

$$\mathbf{1} \,ш\, w = w \,ш\, \mathbf{1} = w, \quad \mathbf{1} * w = w * \mathbf{1} = w \tag{1}$$

for any word w, and

$$x_1 u \,ш\, x_2 v = x_1(u \,ш\, x_2 v) + x_2(x_1 u \,ш\, v), \tag{2}$$

$$z_j u * z_k v = z_j(u * z_k v) + z_k(z_j u * v) + z_{j+k}(u * v) \tag{3}$$