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On the coefficients of power sums of arithmetic progressions

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We investigate the coefficients of the polynomial

$$S^n_{m,r}(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n.$$

We prove that these can be given in terms of Stirling numbers of the first kind and r-Whitney numbers of the second kind. Moreover, we prove a necessary and sufficient condition for the integrity of these coefficients.

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1. Introduction

Let n be a positive integer, and let

$$S_n(\ell) = 1^n + 2^n + \cdots + (\ell - 1)^n$$

be the power sum of the first $\ell - 1$ positive integers. It is well known that $S_n(\ell)$ is strongly related to the Bernoulli polynomials $B_n(x)$ in the following way

$$S_n(\ell) = \frac{1}{n+1}(B_{n+1}(\ell) - B_{n+1}),$$

where the polynomials $B_n(x)$ are defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$$

and $B_n = B_n(0)$ is the nth Bernoulli number.

It is possible to find the explicit coefficients of ℓ in $S_n(\ell)$ [9]:

$$S_n(\ell) = \sum_{i=0}^{n+1} \ell^i \left(\sum_{k=0}^{n} S_2(n,k) S_1(k+1,i) \frac{1}{k+1} \right), \tag{1}$$

where $S_1(n,k)$ and $S_2(n,k)$ are the (signed) Stirling numbers of the first and second kind, respectively.

Recently, Bazsó et al. [1] considered the more general power sum

$$S_{m,r}^n(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m + r)^n,$$

where $m \neq 0$, r are coprime integers. Obviously, $S_{1,0}^n(\ell) = S_n(\ell)$. They, among other things, proved that $S_{m,r}^n(\ell)$ is a polynomial of ℓ with the explicit expression

$$S_{m,r}^n(\ell) = \frac{m^n}{n+1}\left(B_{n+1}\left(\ell + \frac{r}{m}\right) - B_{n+1}\left(\frac{r}{m}\right)\right). \tag{2}$$

In [12], using a different approach, Howard also obtained the above relation via generating functions. Hirschhorn [11] and Chapman [5] deduced a more difficult expression which contains only binomial coefficients and Bernoulli numbers.

For some related diophantine results on $S_{m,r}^n(\ell)$ see [3,10,15,16,2] and the references given there.

Our goal is to give the explicit form of the coefficients of the polynomial $S_{m,r}^n(\ell)$, thus generalizing (1). In this expression the Stirling numbers of the first kind also will appear, but, in place of the Stirling numbers of the second kind a more general class of numbers arises, the so-called r-Whitney numbers introduced by the second author [13].