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Pullbacks of Hermitian Maass lifts

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A R T I C L E I N F O

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We consider pullbacks of hermitian Maass lifts of degree 2 to the submanifold of diagonal matrices. By using these pullbacks, we give an explicit formula for central values of L-functions for $\mathrm{GL}(2) \times \mathrm{GL}(2)$.

1. Introduction

Pullbacks of Siegel Eisenstein series have been studied by Böcherer [1], Garrett [8] and Heim [13]. Pullbacks of hermitian Eisenstein series have been studied by Furusawa [5], Harris [10] and Saha [23]. These pullbacks have been used to study the algebraicity of critical values of certain automorphic L-functions. Moreover, one might consider pullbacks of cusp forms. The (Gan–)Gross–Prasad conjecture [9,6] would relate critical values of certain L-functions and the pullbacks of an automorphic representation of $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$ or one of $\mathrm{U}(n+1)$ to $\mathrm{U}(n)$. For example, in [26,15,7], the pullbacks of an automorphic representation of $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$ for small n were studied. In [27] and [28], Zhang studied the Gan–Gross–Prasad conjecture for $\mathrm{U}(n+1)$ to $\mathrm{U}(n)$ for general n assuming some additional conditions. On the other hand, Ichino [14] gave an explicit

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formula for pullbacks of Saito–Kurokawa lifts, which are Siegel cusp forms of degree 2, in terms of central critical values of L-functions for $\mathrm{SL}_2 \times \mathrm{GL}_2$. Ichino and Ikeda [16] gave an explicit formula for the restriction of hermitian Maass lifts of degree 2 to the Siegel upper half space of degree 2 in terms of central critical values of triple product L-functions. These results may be also regarded as special cases of the Gross–Prasad conjecture. In this paper, we relate pullbacks of hermitian Maass lifts of degree 2 to central values of L-functions for $\mathrm{GL}_2 \times \mathrm{GL}_2$.

Let us describe our results. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with discriminant $-D < 0$. We denote the ideal class group of K by Cl_K and the class number of K by h_K. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by χ. Let κ be a positive integer and $f \in S_{2\kappa+1}(\Gamma_0(D), \chi)$ be a normalized Hecke eigenform. For an integral ideal $\mathfrak{c}$ of K which is prime to D, we denote by $F_\mathfrak{c}$ the hermitian Maass lift of f which satisfies the Maass relation for $\mathfrak{c}$. The lift $F_\mathfrak{c}$ is an automorphic form on the hermitian upper half space $\mathcal{H}_2$ of degree 2 with respect to a certain arithmetic subgroup $\Gamma_K^{(2)}[\mathfrak{c}] \subset \mathrm{U}(2,2)(\mathbb{Q})$. See Section 2 for details. Let $C = N(\mathfrak{c})$ be the ideal norm of $\mathfrak{c}$ and $d(C) = \mathrm{diag}(1, C) \in \mathrm{GL}_2(\mathbb{Q})$. The pullback $F_\mathfrak{c}|_{\mathfrak{H}\times\mathfrak{H}}$ is in $S_{2\kappa+2}(\mathrm{SL}_2(\mathbb{Z})) \otimes S_{2\kappa+2}(d(C)^{-1}\,\mathrm{SL}_2(\mathbb{Z})d(C))$. For each normalized Hecke eigenform $g \in S_{2\kappa+2}(\mathrm{SL}_2(\mathbb{Z}))$, we put $g_C(z) = g(z/C) \in S_{2\kappa+2}(d(C)^{-1}\,\mathrm{SL}_2(\mathbb{Z})d(C))$ and consider the period integral $\langle F_\mathfrak{c}|_{\mathfrak{H}\times\mathfrak{H}}, g \times g_C\rangle$ given by

$$\begin{aligned}&\langle F_\mathfrak{c}|_{\mathfrak{H}\times\mathfrak{H}}, g \times g_C\rangle \\ &= \int\limits_{d(C)^{-1}\,\mathrm{SL}_2(\mathbb{Z})d(C)\backslash\mathfrak{H}} \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}} F_\mathfrak{c}\left(\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right) \overline{g(z_1)g_C(z_2)}\, y_1^{2\kappa} y_2^{2\kappa}\, dz_1 dz_2.\end{aligned}$$

Let $L(s, f \times g)$ and $L(s, f \times g \times \chi)$ be the Rankin–Selberg L-function and its twist given by f and g of degree 4. We put $L_\infty(s) = \Gamma_\mathbb{C}(s + 2\kappa + 1/2)\Gamma_\mathbb{C}(s + 1/2)$ with $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. They satisfy the functional equation

$$L_\infty(s)L(s, f \times g) = -D^{1-2s+2\kappa} a_f(D)^{-2} L_\infty(1-s)L(1-s, f \times g \times \chi),$$

where $a_f(D)$ is the D-th Fourier coefficient of f. Let $L(s, \chi)$ be the Dirichlet L-function associated with χ.

Our main result is as follows.

Theorem 3.2. *The identity*

$$L\left(\frac{1}{2}, f \times g\right) = \frac{L(1,\chi)(4\pi)^{2\kappa+1}}{a_f(D)(2\kappa)!} \cdot \frac{1}{h_K} \sum_{[\mathfrak{c}] \in Cl_K} \frac{\langle F_\mathfrak{c}|_{\mathfrak{H}\times\mathfrak{H}}, g \times g_C\rangle}{\langle g_C, g_C\rangle}$$

holds.