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The 2-adic valuations of differences of Stirling numbers of the second kind

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Let m, n, k and c be positive integers, $\nu_2(k)$ be the 2-adic valuation of k and $S(n,k)$ be the Stirling numbers of the second kind. We show that if $2 \le m \le n$ and c is odd, then $\nu_2(S(c2^{n+1}, 2^m - 1) - S(c2^n, 2^m - 1)) = n + 1$ except when $n = m = 2$ and $c = 1$, in which case $\nu_2(S(8,3) - S(4,3)) = 6$. This solves a conjecture of Lengyel proposed in 2009.

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1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers and let $n, k \in \mathbb{N}$. The *Stirling numbers of the second kind* $S(n, k)$ are defined as the number of ways to partition a set of n elements into exactly k non-empty subsets. There hold the explicit formula $S(n, k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$ and the recurrence relation $S(n, k) = S(n-1, k-1)+kS(n-1, k)$ with initial condition $S(0, 0) = 1$ and $S(n, 0) = 0$ for $n > 0$. Further, the generating functions $(e^x-1)^k = k!\sum_{j=k}^{\infty} S(j, k)\frac{x^j}{j!}$ and $\prod_{i=1}^{k}\frac{1}{1-ix} = \sum_{j=0}^{\infty} S(j+k, k)x^j$ are satisfied. Given a prime p and a positive integer m, there exist unique integers a and n, with $p \nmid a$ and $n \geq 0$, such that $m = ap^n$. The number n is called *p-adic valuation* of m, denoted by $n = v_p(m)$. The study of p-adic valuations of Stirling numbers of the second kind is important in algebraic topology and full with challenging problems (see, for instance, [1–7,10–12,14]). Lengyel [11] conjectured that if $n, k, a, b \in \mathbb{N}$ with $3 \leq k \leq 2^n$, then $\nu_2(S(a2^n, k) - S(b2^n, k)) = n + 1 + \nu_2(a - b) - f(k)$ for some function $f(k)$ which is independent of n (for any sufficiently large n). Lengyel [11] showed that this conjecture is true if $s_2(k) \leq 2$ and $k > 3$. Zhao, Hong and Zhao [14] used Junod's congruence [8] to show this conjecture except when k is a power of 2 minus 1, in which case this conjecture is still kept open so far. It is noted in [14] that the techniques there are not suitable for treating with the remaining case that k is a power of 2 minus 1. Meanwhile, Lengyel [11] suggested another conjecture saying that for any $c, m, n \in \mathbb{N}$ with $c \geq 1$ being odd and $2 \leq m \leq n$, one has $\nu_2(S(c2^{n+1}, 2^m - 1) - S(c2^n, 2^m - 1)) = n + 1$. Note that this latter conjecture is stronger than the former conjecture when $a = 2b = 2c$.

In this paper, we introduce a new method to study the conjectures of Lengyel mentioned above. We will develop a detailed 2-adic analysis to the Stirling numbers of the second kind. The main results of this paper can be stated as follows.

Theorem 1.1. *Let $a, b, n \in \mathbb{N}$ with $a > b \geq 1$. For $r \in \{2, 3\}$, define $T_r := \nu_2(S(a2^n, 2^r - 1) - S(b2^n, 2^r - 1))$. If $n \geq r$, then $T_r = n+1+\nu_2(a-b)$ if $b2^n > n+r+\nu_2(a-b)$, $T_r > n+1+\nu_2(a-b)$ if $b2^n = n+r+\nu_2(a-b)$, and $T_r = b2^n-1$ if $b2^n < n+r+\nu_2(a-b)$.*

Theorem 1.2. *Let $c, m, n \in \mathbb{N}$ with $c \geq 1$ being odd and $2 \leq m \leq n$. Then $\nu_2(S(c2^{n+1}, 2^m - 1) - S(c2^n, 2^m - 1)) = n + 1$ except when $n = m = 2$ and $c = 1$, in which case one has $\nu_2(S(8, 3) - S(4, 3)) = 6$.*

Evidently, by Theorem 1.1, we know that the first conjecture of Lengyel is true for the cases that $k = 3$ and 7 and sufficiently large n, but its truth still keeps open when k is a power of 2 minus 1 and no less than 15. By Theorem 1.2 one knows that the second conjecture of Lengyel holds except for the case that $n = m = 2$ and $c = 1$, in which case it is not true.

This paper is organized as follows. In Section 2, we recall some known results and show also several new results that are needed in the proofs of Theorems 1.1 and 1.2. The proofs of Theorems 1.1 and 1.2 are given in Section 3. The new key ingredients in