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On the divisibility of the truncated hypergeometric function ${}_3F_2$ ☆

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ABSTRACT

Suppose that p is an odd prime and α, β are prime to p. We prove that p^2 divides the truncated hypergeometric function

$${}_3F_2\left[\begin{matrix}\alpha & \beta & 1-\alpha-\beta \\ & 1 & 1\end{matrix}\middle| 1\right]_p$$

provided $\langle\alpha\rangle_p + \langle\beta\rangle_p \le p$, where $\langle\alpha\rangle_p$ denotes the least non-negative residue of α modulo p.

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The truncated hypergeometric function ${}_{q+1}F_q\left[\begin{matrix}\alpha_1 & \alpha_2 & \dots & \alpha_{q+1}\\ & \beta_1 & \dots & \beta_q\end{matrix}\middle| x\right]_n$ is given by

$$ {}_pF_q\left[\begin{matrix}\alpha_1 & \alpha_2 & \dots & \alpha_p\\ \beta_1 & \beta_2 & \dots & \beta_q\end{matrix}\middle| x\right]_n = \sum_{k=0}^{n-1}\frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_q)_k}\cdot\frac{x^k}{k!}. $$

where

$$ (\alpha)_k = \begin{cases}\alpha(\alpha+1)\cdots(\alpha+k-1), & \text{if } k\ge 1,\\ 1, & \text{if } k=0.\end{cases} $$

In [7] and [8], with the help of the Gross–Koblitz formula, E. Mortenson studied the arithmetical properties of ${}_{q+1}F_q\left[\begin{matrix}m_1/d_1 & m_2/d_2 & \dots & m_{q+1}/d_{q+1}\\ & 1 & \dots & 1\end{matrix}\middle| 1\right]_p$, where $1 \le m_i < d_i$ and p is a prime with $p \equiv 1 \pmod{[d_1,\dots,d_{q+1}]}$. He showed that ${}_{q+1}F_q\left[\begin{matrix}m_1/d_1 & m_2/d_2 & \dots & m_{q+1}/d_{q+1}\\ & 1 & \dots & 1\end{matrix}\middle| 1\right]_p$ modulo p^2 can be represented as the sum of products of some Jacobi sums. As the applications, Mortenson resolved several conjectures of Rodriguez-Villegas. For example, he proved that

$$ {}_2F_1\left[\begin{matrix}1/2 & 1/2\\ & 1\end{matrix}\middle| 1\right]_p \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, \qquad {}_2F_1\left[\begin{matrix}1/3 & 2/3\\ & 1\end{matrix}\middle| 1\right]_p \equiv \left(\frac{-3}{p}\right) \pmod{p^2}, $$
$$ {}_2F_1\left[\begin{matrix}1/4 & 3/4\\ & 1\end{matrix}\middle| 1\right]_p \equiv \left(\frac{-2}{p}\right) \pmod{p^2}, \qquad {}_2F_1\left[\begin{matrix}1/6 & 5/6\\ & 1\end{matrix}\middle| 1\right]_p \equiv \left(\frac{-1}{p}\right) \pmod{p^2}, $$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol modulo p.

Subsequently, using the Zeilberger algorithm, Z.-W. Sun [15] gave the elementary proofs for the above four congruences. Furthermore, in [13], Z.-H. Sun considered ${}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle| x\right]_p$ modulo p^2. In fact, he proved that

$$ {}_2F_1\left[\begin{matrix}\alpha & 1-\alpha\\ & 1\end{matrix}\middle| 1\right]_p \equiv (-1)^{\langle\alpha\rangle_p - 1} \pmod{p^2}, $$

where $\langle\alpha\rangle_p$ denotes the least non-negative residue of α modulo p.

In [8], Mortenson also established some interesting congruences for truncated ${}_3F_2$ modulo p^2. For example, he proved that

$$ {}_3F_2\left[\begin{matrix}1/2 & 1/2 & 1/2\\ & 1 & 1\end{matrix}\middle| 1\right]_p \equiv a(p) \pmod{p^2}, $$

where $a(n)$ arises from the newform

$$ \sum_{n=1}^{\infty} a(n)q^n = \eta(4z)^6 \in S_3(\Gamma_0(16), (\tfrac{-4}{d})) $$

and $\eta(z)$ is the Dedekind η-function. Notice that according to [10, 14.2], we have