Journal of Number Theory 153 (2015) 354–363

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

A diophantine problem from calculus

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ARTICLE INFO

Article history:
Received 10 May 2014
Received in revised form 12 January 2015
Accepted 22 January 2015
Available online 5 March 2015
Communicated by David Goss

MSC:
11C08
11D25

Keywords:
Nice polynomial
Quartic polynomial
Quintic polynomial
Sextic polynomial

ABSTRACT

A univariate polynomial $f(x)$ is said to be nice if all of its coefficients as well as all of the roots of both $f(x)$ and its derivative $f'(x)$ are integers. The known examples of nice polynomials with distinct roots are limited to quadratic polynomials, cubic polynomials, symmetric quartic polynomials and, up to equivalence, only a finite number of nonsymmetric quartic polynomials and one quintic polynomial. In this paper we find parametrized families of nice nonsymmetric quartic polynomials with distinct roots, as well as infinitely many nice quintic and sextic polynomials with distinct roots.

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1. Introduction

A univariate polynomial $f(x)$ is said to be nice if all of its coefficients as well as all of the roots of both $f(x)$ and its derivative $f'(x)$ are integers. While nice polynomials of any arbitrary degree are known, they have repeated roots, a simple example being $x^{n-1}(x-n)$. This paper is concerned with nice polynomials that have distinct roots.

It is easy to observe that if $f(x)$ is a nice polynomial of degree d and a, b, c are arbitrary integers such that $a \neq 0$, $c \neq 0$, then $g(x) = a^d cf((x+b)/a)$ is also a nice

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http://dx.doi.org/10.1016/j.jnt.2015.01.010
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polynomial of degree d. This observation seems to have been made first of all by Caldwell [3]. In fact, the polynomial $g(x)$ is at times a nice polynomial even for certain noninteger rational values of a, b, c. Thus, given a single nice polynomial $f(x)$ of degree d, we readily obtain infinitely many nice polynomials of degree d. Two nice polynomials that are obtained in this manner from the same nice polynomial will be considered equivalent. More precisely, two nice polynomials $f_1(x)$ and $f_2(x)$ of degree d will be considered equivalent if there exist rational numbers r, s, t such that $rs \neq 0$ and $f_1(x) = rf_2(sx+t)$. When we refer to distinct nice polynomials, we will mean nice polynomials that are not equivalent.

A complete determination of all nice quadratic and cubic polynomials with distinct roots has already been done [1,2,7]. Nice quartic polynomials with distinct roots were first found by Caldwell [3] but this paper has remained inaccessible to me. It appears from the detailed discussion in [6, pp. 21–23] that while all nice symmetric quartic polynomials have been found, only finitely many distinct nice nonsymmetric quartic polynomials are known. Further, up to equivalence, only one numerical example of a nice quintic polynomial has been found [6, p. 23]. No nice sextic polynomials have been found till now.

In this paper we find nice nonsymmetric quartic polynomials whose coefficients are given in parametric terms and whose roots are all distinct. This yields infinitely many distinct nice nonsymmetric quartic polynomials. We also find infinitely many distinct nice quintic and sextic polynomials whose roots are all distinct.

It is readily seen that if $f(x)$ is an nth degree polynomial with n rational roots such that all of the $n-1$ roots of $f'(x)$ are also rational, then for a suitably chosen integer value of h, the polynomial $h^n f(X/h)$ is such that its coefficients are integers and all of the roots of both $h^n f(X/h)$ and its derivative are integers. Thus to obtain nice polynomials, it suffices to construct polynomials $f(x)$ such that all the roots of both $f(x)$ and its derivative $f'(x)$ are rational.

2. Nice quartic polynomials with distinct roots

It will be recalled that a polynomial $f(x)$ is symmetric if there exists a number C such that $f(C+x) = f(C-x)$ for all values of x. If $f(x)$ is a nice polynomial, so is $f(x+t)$ where t is an arbitrary integer. By a suitable choice of t, the centre C of a nice symmetric polynomial can be taken as 0. Thus every nice symmetric polynomial is equivalent to a symmetric polynomial with centre 0 and leading coefficient 1. As the variable x occurs only in even degrees in such symmetric polynomials, when searching for nice symmetric quartic polynomials, it suffices to consider the symmetric polynomial $f(x) = (x^2 - a^2)(x^2 - b^2)$. Its derivative $f'(x) = 4x^3 - 2(a^2 + b^2)x$ will have all rational roots if $a^2 + b^2 = 2r^2$. Since all solutions of this diophantine equation are readily found, we easily obtain all nice symmetric quartic polynomials.

We will now obtain nice nonsymmetric quartic polynomials. Let us consider the quartic polynomial,

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