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Sets of positive integers closed under product and the number of decimal digits

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A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$ with $\ell(n)$ the number of digits of n written in decimal expansion. In this note, we compute the smallest digital semigroup containing a set of positive integers. For this, we establish a connection between the digital semigroups and a class of numerical semigroups called LD-semigroups.

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1. Introduction

Given a positive integer n, we denote by $\ell(n)$ the number of digits of n write in decimal expansion. For example $\ell(137) = 3$ and $\ell(2335) = 4$.

Let $\mathbb{N}$ be the set of nonnegative integers. A digital semigroup D is a subsemigroup of $(\mathbb{N}\backslash\{0\}, \cdot)$ such that if $d \in D$ then $\{x \in \mathbb{N}\backslash\{0\} \mid \ell(x) = \ell(d)\} \subseteq D$. Our main goal in this paper is to study the digital semigroups. We are interested in determining the smallest digital semigroup containing a set of positive integers. For this, we will establish a connection between the digital semigroups and a class of numerical semigroups which we will call LD-semigroups.

A numerical semigroup is a submonoid S of $(\mathbb{N}, +)$ such that $\mathbb{N}\backslash S$ is finite (the cardinality of $\mathbb{N}\backslash S$, $\mathrm{g}(S)$, is the gender of S). Given, A a subset of $\mathbb{N}\backslash\{0\}$, we denote by $L(A) = \{\ell(a) \mid a \in A\}$. We prove that if D a digital semigroup then $L(D) \cup \{0\}$ is a numerical semigroup. A numerical semigroup S is called LD-semigroup if there exist a digital semigroup D such that $S = L(D) \cup \{0\}$.

Denote by $\mathcal{D}$ (respectively $\mathcal{L}$) the set of all digital semigroups (respectively LD-semigroups). We see that the map $\varphi : \mathcal{D} \longrightarrow \mathcal{L}$ defined by $\varphi(D) = L(D) \cup \{0\}$ is bijective and its inverse is the map $\theta : \mathcal{L} \longrightarrow \mathcal{D}$ with $\theta(S) = \{a \in \mathbb{N}\backslash\{0\} \mid \ell(a) \in S\}$. From this it easily follows that if D is a digital semigroup then $\mathbb{N}\backslash D$ is finite.

We characterize the LD-semigroups in the following way: a numerical semigroup S is an LD-semigroup if and only if $a + b - 1 \in S$ for all $a, b \in S\backslash\{0\}$ (Theorem 4). This fact allows us prove that $\mathcal{L}$ is a Frobenius variety (see [4]). The greatest integer that does not belong to a numerical semigroup S is called the Frobenius number of S and it is denoted here by $\mathrm{F}(S)$ (see [3]). A Frobenius variety is a nonempty set $\mathcal{V}$ of numerical semigroups fulfilling the following conditions:

(1) if S and T are in $\mathcal{V}$, then so is $S \cap T$;
(2) if S is in $\mathcal{V}$ and it is not equal to $\mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is in $\mathcal{V}$.

This fact together with the results presented in [4] allows us to arrange the elements of $\mathcal{L}$ in a tree. We characterize the sons of any vertex of this tree and this will enable us to recursively construct the set $\mathcal{L}$ and consequently the set $\mathcal{D}$.

Given a set of positive integers X we denote by $\mathcal{D}(X)$ (respectively $\mathcal{L}(X)$) the smallest (with respect to the set inclusion order) digital semigroup containing X (respectively LD-semigroup). We prove that if X is a set of positive integers and S the smallest LD-semigroup containing $L(X)$ then $\theta(S)$ is the smallest digital semigroup containing X. As a first consequence of this we get that $\mathcal{D} = \{\mathcal{D}(X) \mid \text{X is a nonempty finite subset of } \mathbb{N}\backslash\{0\}\}$ whence every digital semigroup can be described from a finite number of terms.

Given a finite set of positive integers X we describe an algorithmic procedure for computing the smallest LD-semigroup that contains X. As a consequence we have an algorithm that computes the smallest digital semigroup containing a finite set of positive integers.