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Some new continued fraction approximation of Euler's constant

Dawei Lu *, Lixin Song, Yang Yu

School of Mathematical Sciences, Dalian University of Technology, Dalian 116023, China

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In this paper, using continued fraction, some quicker classes of sequences convergent to Euler's constant are provided. Finally, for demonstrating the superiority of our new convergent sequences over DeTemple's sequence, Vernescu's sequence and Mortici's sequences, some numerical computations are also given.

* Corresponding author.
E-mail addresses: ludawei_dlut@163.com (D. Lu), lxsong@dlut.edu.cn (L. Song), yuyang_dut@163.com (Y. Yu).

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1. Introduction

An important concern in the theory of mathematical constants is to define some new sequences which have higher convergent speed towards some fundamental constants. Those constants and new sequences play an important role in many fields of mathematics and nature science, such as special functions, theory of probability, physics, applied statistics, number theory, and analysis.

It is well known that one of the most useful convergent sequences is

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n, \tag{1.1}$$

whose limit is known as Euler's constant, denoted by

$$\gamma = 0.577215\ldots.$$

So far, many researchers have devoted great efforts and achieved much in the area of improving the convergent rate of the sequence $(\gamma_n)_{n\geq 1}$. Among them, there are many inspiring achievements. For example, in [12–14,16], the estimate

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n} \tag{1.2}$$

was established with interesting geometric interpretations. In [1,2], a faster convergent sequence $(D_n)_{n\geq 1}$ to γ was introduced, which is defined as

$$D_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right). \tag{1.3}$$

DeTemple also concluded that the speed of the new sequence to γ is the same as the speed of convergence n^{-2}, since

$$\frac{1}{24(n+1)^2} < D_n - \gamma < \frac{1}{24n^2}. \tag{1.4}$$

In [15], Vernescu presented a new modification,

$$V_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \ln n, \tag{1.5}$$

and the estimate was provided as

$$\frac{1}{12(n+1)^2} < \gamma - V_n < \frac{1}{12n^2}. \tag{1.6}$$

In both (1.3) and (1.5), only slight modifications are made to Euler's sequence (1.1), but the convergent rates are significantly improved from n^{-1} to n^{-2}.