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On simultaneous approximation to powers of a real number by rational numbers

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ABSTRACT

In this paper, we obtain an upper bound of the uniform exponent of simultaneous approximation for certain powers of a real number. This is a partial generalization of a result of Lozier–Roy.

1. Introduction

This paper treats the problem of uniform simultaneous approximation by rational numbers to a certain set of real numbers, which was initiated by Davenport–Schmidt [1] in the study of approximation to real numbers by algebraic integers of bounded degree.

First we explain their result (with improvements by Laurent [2] and Roy [4,5]), in the form explained in [3]. Fix a positive integer n and an element $\Xi = (\xi_0, ..., \xi_n) \in \mathbb{R}^{n+1}$ with $\xi_0 \neq 0$. We say that a real number λ is a uniform exponent of approximation to Ξ by rational numbers if there exists a constant $c > 0$ such that the system of inequalities

$$\max_{0\le i\le n}\left(|x_i|\right) \le X, \qquad \max_{0\le i\le n}\left(|x_0\xi_i - x_i\xi_0|\right) \le cX^{-\lambda}$$

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admits a non-zero solution $\mathbf{x} = (x_0, ..., x_n) \in \mathbb{Z}^{n+1}$ for each real number $X > 1$. Let $\lambda(\Xi)$ be the supremum of such exponents λ. Then the following theorem is known:

Theorem 1.1 *(Davenport–Schmidt, Laurent, Roy). Let $n \geq 2$ be an integer and let $\xi \in \mathbb{R}$ be a real number such that $1, \xi, ..., \xi^n$ are linearly independent over $\mathbb{Q}$. Then, if we put $\Xi := (1, \xi, ..., \xi^n)$, we have $\lambda(\Xi) \leq 1/\gamma \cong 0.618$ if $n = 2$, $\lambda(\Xi) \leq (1+2\gamma-\sqrt{1+4\gamma^2})/2 \cong 0.4245$ if $n = 3$ and $\lambda(\Xi) \leq \lceil \frac{n}{2} \rceil^{-1}$ if $n \geq 4$, where $\gamma := (1 + \sqrt{5})/2$ is the golden ratio and $\lceil \frac{n}{2} \rceil$ denotes the smallest integer greater than or equal to $\frac{n}{2}$.*

This bound of $\lambda(\Xi)$ is optimal in the case $n = 2$ in the sense that the supremum of $\lambda(\Xi)$'s for ξ's in the theorem is equal to $1/\gamma \cong 0.618$ (Roy [4]), but the problem to find the optimal bound is open for $n \geq 3$.

Lozier–Roy [3] studied a similar problem for the triples of the form $(1, \xi, \xi^3)$ and obtained the following result:

Theorem 1.2 *(Lozier–Roy). Let $\xi \in \mathbb{R}$ be a real number such that 1, ξ, ξ^3 are linearly independent over $\mathbb{Q}$. Then, if we put $\Xi := (1, \xi, \xi^3)$, we have $\lambda(\Xi) \leq 2(9 + \sqrt{11})/35 \cong 0.7038$.*

We explain the method of their proof. First they associate a sequence of minimal points $\{\mathbf{x}_i\}_{i\geq 1}$ ($\mathbf{x}_i := (x_{i,0}, x_{i,1}, x_{i,2}) \in \mathbb{Z}^3$) as in [1]. Then the set I of indices i such that $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are linearly independent is an infinite set. If we put $\varphi(\mathbf{x}) := x_0^2 x_2 - x_1^3$, $\varphi(\mathbf{x}_i)$ is nonzero for large $i \in I$ by algebraic reason and it is estimated from above by analytic reason. Using these, they control the growth of $X_i := \max(|x_{i,0}|, |x_{i,1}|, |x_{i,2}|)$ ($i \in I$) and get the first estimate $\lambda(\Xi) \leq \sqrt{3} - 1 \cong 0.732$. To get better estimates, they find nice homogeneous polynomials of the form $P(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}[\mathbf{x}, \mathbf{y}]$ (where $\mathbf{x} = (x_0, x_1, x_2)$, $\mathbf{y} = (y_0, y_1, y_2)$ are triples of indeterminates) such that, if we denote the next element of $i \in I$ in I by $n(i)$, the integer $P(\mathbf{x}_i, \mathbf{x}_{n(i)})$ is estimated from above by analytic reason and that the vanishing of this integer implies certain divisibility property of the value of a related polynomial at $\mathbf{x}_i$ or $(\mathbf{x}_i, \mathbf{x}_{n(i)})$. Then, by proving the non-vanishing of $P(\mathbf{x}_i, \mathbf{x}_{n(i)})$, they get a certain estimate from below by the divisibility property, and by combining this with the estimate from above they get better control on the growth of X_i ($i \in I$) and get better estimates of $\lambda(\Xi)$. More precisely, by using the polynomial $F(\mathbf{x}, \mathbf{y})$ introduced in [3, §3], they get the second estimate $\lambda(\Xi) \leq 5/7 \cong 0.714$, and by using the polynomials $D^{(2)}$, $D^{(3)}$, $D^{(6)}$ introduced in [3, §4], they get the final estimate $\lambda(\Xi) \leq 2(9 + \sqrt{11})/35 \cong 0.7038$ in the above theorem. Note that this bound will not yet be optimal.

In this paper, we study a similar problem for the triples of the form $(1, \xi^l, \xi^k)$ for integers $1 \leq l < k$ (the case $k \geq 4$ is new). By the same argument as above using the polynomial $\varphi(\mathbf{x}) = x_0^{k-l} x_2^l - x_1^k$, we get the first estimate $\lambda(\Xi) \leq \lambda_0$, where λ_0 is the larger root of $\lambda^2 + (k-1)\lambda - (k-1)$. Then, by using a suitable polynomial $G(\mathbf{x}, \mathbf{y})$, which plays a role of the polynomial $F(\mathbf{x}, \mathbf{y})$ of Lozier–Roy, we get the following estimate for odd k, which is the main theorem in this paper.