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Self-reciprocal functions, powers of the Riemann zeta function and modular-type transformations

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ABSTRACT

Integrals containing the first power of the Riemann Ξ-function as part of the integrand that lead to modular-type transformations have been previously studied by Ramanujan, Hardy, Koshlyakov, Ferrar and others. An integral containing the square of the Riemann Ξ-function and involving an extra parameter z, whose type naturally extends that of the aforementioned integrals, was studied by Ramanujan. This integral implicitly involves squaring of the functional equation of $\zeta(s)$. A unifying procedure to analyze general integrals of this type is studied here along with the interesting modular transformations that they generate. This also includes generalization of some transformations of Koshlyakov involving a series containing the modified Bessel function $K_0(x)$.

1. Introduction

The Riemann ξ-function $\xi(s)$ is defined by

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{1.1}$$

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where $\Gamma(s)$ and $\zeta(s)$ are the gamma and the Riemann zeta functions respectively. $\xi(s)$ is an entire function of s. On the critical line $\operatorname{Re} s = \frac{1}{2}$, it is denoted by

$$\Xi(t) := \xi\left(\frac{1}{2} + it\right), \tag{1.2}$$

and is called the Riemann Ξ-function. It is well-known that $\Xi(t)$ is an even function of t and is real-valued for a real t. The functional equation of $\zeta(s)$, given by

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{1.3}$$

can also be written in terms of the ξ-function as [30, p. 16]

$$\xi(s) = \xi(1-s).$$

The integral evaluation

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t/2)}{1+t^2}\cos\left(\frac{1}{2}t\log\alpha\right)dt = \sqrt{\alpha}\left(\frac{1}{2\alpha} - \sum_{n=1}^\infty e^{-\pi\alpha^2 n^2}\right) \tag{1.4}$$

is well-known [30, p. 36].

The goal of this paper is to study, in a unified manner, a variety of integrals of the form

$$I(f, z; \alpha) = \int_0^\infty f\left(z, \frac{t}{2}\right)\Xi\left(\frac{t-iz}{2}\right)\Xi\left(\frac{t+iz}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt, \tag{1.5}$$

where $f(z,t)$ is an even function of t of the form

$$f(z,t) = \phi(z, it)\phi(z, -it), \tag{1.6}$$

with ϕ analytic as a function of $t \in \mathbb{R}$ and $z \in \mathbb{C}$. The integral (1.5) extends

$$I(f; \alpha) = \int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt, \tag{1.7}$$

with

$$f(t) = \varphi(it)\varphi(-it), \tag{1.8}$$

and φ is analytic in t, of which the integral in (1.4) is a special case. Other particular examples of the integral (1.7) were studied by Ramanujan, Hardy, Koshlyakov and Ferrar