Journal of Number Theory 147 (2015) 490–498

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Euler sums of hyperharmonic numbers

Ayhan Dil [a,*], Khristo N. Boyadzhiev [b]

[a] Department of Mathematics, Akdeniz University, 07058-Antalya, Turkey
[b] Department of Mathematics and Statistics, Ohio Northern University, Ada, OH 45810, USA

ARTICLE INFO

Article history:
Received 9 November 2013
Received in revised form 8 May 2014
Accepted 14 July 2014
Available online 16 September 2014
Communicated by David Goss

MSC:
11B73
11B83
11M99

Keywords:
Riemann zeta function
Hurwitz zeta function
Euler sums
Harmonic and hyperharmonic numbers
Stirling numbers
Beta function

ABSTRACT

The hyperharmonic numbers $h_n^{(r)}$ are defined by means of the classical harmonic numbers. We show that the Euler-type sums with hyperharmonic numbers:

$$\sigma(r,m)=\sum_{n=1}^{\infty}\frac{h_n^{(r)}}{n^m}$$

can be expressed in terms of series of Hurwitz zeta function values. This is a generalization of a result of Mező and Dil (2010) [7]. We also provide an explicit evaluation of $\sigma(r,m)$ in a closed form in terms of zeta values and Stirling numbers of the first kind. Furthermore, we evaluate several other series involving hyperharmonic numbers.

1. Introduction

In this paper we are interested in Euler-type sums with hyperharmonic numbers $\sigma(r,m)$. Such series could be of interest in analytic number theory. We will show that

* Corresponding author.
E-mail addresses: adil@akdeniz.edu.tr (A. Dil), k-boyadzhiev@onu.edu (K.N. Boyadzhiev).

http://dx.doi.org/10.1016/j.jnt.2014.07.018

these sums are related to the values of the Riemann zeta function. In [7] the authors considered the case $r = 1$. Here we extend this result to $r > 1$.

In the second section we express $\sigma(r, m)$ as a special series involving zeta values. In the third section we evaluate $\sigma(r, m)$ as a finite sum including Stirling numbers of the first kind, zeta values, and values of the digamma (psi) function.

In the last fourth section we use certain integral representations to evaluate several series with hyperharmonic numbers. For example,

$$\sum_{n=1}^{\infty} \frac{h_n^{(r)}}{n(n+1)\dots(n+r)} = \frac{\pi^2}{6r!}$$

and

$$\sum_{n=1}^{\infty} h_n^{(r)} B(r+1, n+1) = 1$$

where $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $B(r, n)$ is the Beta function.

1.1. Definitions and notation

The n-th harmonic number is defined by

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \quad \bigl(n \in \mathbb{N} := \{1, 2, 3, \dots\}\bigr), \tag{1}$$

where the empty sum H_0 is conventionally understood to be zero.

Starting with $h_n^{(0)} = \frac{1}{n}$ $(n \in \mathbb{N})$, the n-th hyperharmonic number $h_n^{(r)}$ of order r is defined by (see [4], see also [7]):

$$h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)} \quad (r \in \mathbb{N}). \tag{2}$$

It is easy to see that $h_n^{(1)} := H_n$ $(n \in \mathbb{N})$.

These numbers can be expressed in terms of binomial coefficients and ordinary harmonic numbers (see [4,7]):

$$h_n^{(r)} = \binom{n+r-1}{r-1} (H_{n+r-1} - H_{r-1}). \tag{3}$$

The well-known generating functions of the harmonic and hyperharmonic numbers are given as

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x} \tag{4}$$