Journal of Number Theory 147 (2015) 508–514

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the conditional infiniteness of primitive weird numbers

Giuseppe Melfi

University of Neuchâtel, Faculty of Economics and Business, Rue A.-L. Breguet, 2, CH-2000 Neuchâtel, Switzerland

ARTICLE INFO

Article history:
Received 30 January 2014
Received in revised form 8 July 2014
Accepted 28 August 2014
Available online 16 September 2014
Communicated by F. Pellarin

MSC:
11A25
11B83

Keywords:
Abundant numbers
Semiperfect numbers
Almost perfect numbers
Sum-of-divisor function
Erdős problems

ABSTRACT

Text. A weird number is a number n for which $\sigma(n) > 2n$ and such that n is not a sum of distinct proper divisors of n. In this paper we prove that $n = 2^k pq$ is weird for a quite large set of primes p and q. In particular this gives an algorithm to generate very large primitive weird numbers, i.e., weird numbers that are not multiple of other weird numbers. Assuming classical conjectures on the gaps between consecutive primes, this also would prove that there are infinitely many primitive weird numbers, a question raised by Benkoski and Erdős in 1974.

Video. For a video summary of this paper, please visit http://youtu.be/OS93l3a_Mjo.

1. Introduction

Let n be a positive integer. Let $\sigma(n)$ denote the sum of divisors of n. If $\sigma(n) > 2n$, then n is called abundant. If n can be expressed as a sum of distinct proper divisors of n, then n is called semiperfect (or sometimes also pseudo-perfect). A weird number is an abundant number that is not semiperfect, i.e., that cannot be expressed as a sum of distinct proper divisors of n.

The term *weird* has been introduced in 1972 by Benkoski [2,3]. In his joint paper with Erdős [4], several results on weird numbers and related questions are proved. In

http://dx.doi.org/10.1016/j.jnt.2014.07.024

particular they proved that there are infinitely many weird numbers, and that indeed, the set of weird numbers has positive asymptotic density.

If n is weird and $p > \sigma(n)$, then np is weird (see for example [6, p. 332]). In this paper we study the properties of *primitive* weird numbers, so those weird numbers that are not a multiple of other weird numbers.

An open problem is to determine whether infinitely many primitive weird numbers exist. Benkoski and Erdős stated this problem as an open question rather than as a conjecture, and in recent literature, as for example in [8, p. 77] and [12, p. 43], this problem still appears as an open question. Computational approaches allowed to provide a list of primitive weird numbers not exceeding $1.8 \cdot 10^9$ [13, Sequence A002975]. Kravitz [10] proved in 1976 that for a prime p, with $p > 2^k$, if $q = [2^k p - (p+1)]/[(p+1) - 2^k]$ is prime, then $2^{k-1}pq$ is a primitive weird number, and found eleven weird numbers among which a 53-digit number that has been for many years the largest known primitive weird number, until Klyve [9] announced a 226-digit weird number in 2013.

In this paper we prove the following theorem:

Theorem 1. *Let k be a positive integer and let a and b be positive odd integers such that $p = 2^{k+2} - a$ and $q = 2^{k+2} + b$ are primes. If $b + 3 < a < 2^{(k-1)/2}$ then $n = 2^k pq$ is a primitive weird number.*

Primitive weird numbers of the form $2^k pq$ appear to be quite common: among the first 160 primitive weird numbers, 116 are of this form. The idea of the theorem is to search for such primitive weird numbers with p and q in a close neighborhood of 2^{k+2}. The condition $b + 3 < a$ ensures that n is a primitive abundant number, and the sizes of a and b, both bounded by $2^{(k-1)/2}$, are such that all relevant divisors of n are in a close neighborhood of a multiple of 2^{k+2}. In this way there are quite large intervals of consecutive positive integers that cannot be reached by a sum of distinct proper divisors of n.

Note also that if k is too small, there are no values of a and b that fit the conditions of the theorem. The triple (a, b, k) that fits the conditions of the theorem and yields the least positive integer is $(5, 1, 6)$ for $2^6(2^8 - 5)(2^8 + 1) = 4\,128\,448$, that is the 32nd primitive weird number, and there are no other triples with $k \leq 7$.

In 1980 Pajunen [11] characterized primitive weird numbers of the form $2^k pq$ in terms of structure properties, but as far as we know no computational or analytical implications have been investigated since then.

Let $\{p_n\}_{n\in\mathbb{N}}$ denote the increasing sequence of primes. If we assume that for sufficiently large n, $p_{n+1} - p_n < 0.1 p_n^{1/2}$, Theorem 1 implies that there are infinitely many primitive weird numbers of the form $2^k pq$. Indeed, infinitely many of such primes should exist: on a hand, Cramér's conjecture [5] claims that $p_{n+1} - p_n = O((\log p_n)^2)$; on the other hand a much weaker conjecture as for example Gonek's conjecture [7, p. 398], claiming that given an arbitrary $\varepsilon > 0$, $p_{n+1} - p_n < p_n^{\varepsilon}$ for sufficiently large n, is sufficient for the existence of primes postulated in the theorem.