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On small bases which admit countably many expansions

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ABSTRACT

Let $q \in (1,2)$ and $x \in [0, \frac{1}{q-1}]$. We say that a sequence $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ is an expansion of x in base q (or a q-expansion) if

$$x = \sum_{i=1}^{\infty} \epsilon_i q^{-i}.$$

Let $\mathcal{B}_{\aleph_0}$ denote the set of q for which there exists x with exactly $\aleph_0$ expansions in base q. In [5] it was shown that $\min \mathcal{B}_{\aleph_0} = \frac{1+\sqrt{5}}{2}$. In this paper we show that the smallest element of $\mathcal{B}_{\aleph_0}$ strictly greater than $\frac{1+\sqrt{5}}{2}$ is $q_{\aleph_0} \approx 1.64541$, the appropriate root of $x^6 = x^4 + x^3 + 2x^2 + x + 1$. This leads to a full dichotomy for the number of possible q-expansions for $q \in (\frac{1+\sqrt{5}}{2}, q_{\aleph_0})$. We also prove some general results regarding $\mathcal{B}_{\aleph_0} \cap [\frac{1+\sqrt{5}}{2}, q_f]$, where $q_f \approx 1.75488$ is the appropriate root of $x^3 = 2x^2 - x + 1$. Moreover, the techniques developed in this paper imply that if $x \in [0, \frac{1}{q-1}]$ has uncountably many q-expansions then the set of q-expansions for x has cardinality equal to that of the continuum, this proves that the continuum hypothesis holds when restricted to this specific case.

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1. Introduction

Let $q \in (1,2)$ and $I_q = [0, \frac{1}{q-1}]$. Each $x \in I_q$ has an expansion of the form

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{q^i}, \tag{1.1}$$

for some $(\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$. We call such a sequence a *q-expansion* of x, when (1.1) holds we will adopt the notation $x = (\epsilon_1, \epsilon_2, \ldots)_q$. Expansions in non-integer bases were pioneered in the papers of Rényi [11] and Parry [10].

Given $x \in I_q$ we denote the set of q-expansions of x by $\Sigma_q(x)$, i.e.,

$$\Sigma_q(x) = \left\{ (\epsilon_i)_{i=1}^{\infty} \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{q^i} = x \right\}.$$

The endpoints of I_q always have a unique q-expansion, typically an element of $(0, \frac{1}{q-1})$ will have a nonunique q-expansion. In [7] it was shown that for $q \in (1, \frac{1+\sqrt{5}}{2})$ the set $\Sigma_q(x)$ is uncountable for all $x \in (0, \frac{1}{q-1})$. When $q = \frac{1+\sqrt{5}}{2}$ it was shown in [15] that every $x \in (0, \frac{1}{q-1})$ has uncountably many q-expansions unless $x = \frac{(1+\sqrt{5})n}{2} \bmod 1$, for some $n \in \mathbb{Z}$, in which case $\Sigma_q(x)$ is infinite countable. In [12] it was shown that for $q \in (\frac{1+\sqrt{5}}{2}, 2)$ the set $\Sigma_q(x)$ is uncountable for almost every $x \in (0, \frac{1}{q-1})$. Furthermore, if $q \in (\frac{1+\sqrt{5}}{2}, 2)$ then it was shown in [4] that there always exists $x \in (0, \frac{1}{q-1})$ with a unique q-expansion.

In this paper we will be interested in the set of $q \in (1,2)$ for which there exists $x \in (0, \frac{1}{q-1})$ satisfying $\operatorname{card} \Sigma_q(x) = \aleph_0$. More specifically, we will be interested in the set

$$\mathcal{B}_{\aleph_0} := \left\{ q \in (1,2) \;\middle|\; \text{there exists } x \in \left(0, \frac{1}{q-1}\right) \text{ satisfying } \operatorname{card} \Sigma_q(x) = \aleph_0 \right\}.$$

In [5] it was shown that $\min \mathcal{B}_{\aleph_0} = \frac{1+\sqrt{5}}{2}$. We can define $\mathcal{B}_k$ in an analogous way for all $k \geq 1$. It was first shown in [6] that $\mathcal{B}_k \neq \emptyset$ for all $k \geq 2$, this was later improved upon in [14] where it was shown that for each $k \in \mathbb{N}$ there exists $\gamma_k > 0$ such that $(2-\gamma_k, 2) \subset \mathcal{B}_j$ for all $1 \leq j \leq k$. Combining the results stated in [14] and [3] the following theorem is shown to hold.

Theorem 1.1.

(1) *The smallest element of* $\mathcal{B}_2$ *is*

$$q_2 \approx 1.71064,$$

the appropriate root of $x^4 = 2x^2 + x + 1$.