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Proof of a conjecture of Mircea Merca

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A R T I C L E I N F O

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A B S T R A C T

We prove that, for any prime p and positive integer r with $p^r > 2$, the number of multinomial coefficients such that

$$\binom{k}{k_1, k_2, \dots, k_n} = p^r, \quad \text{and} \quad k_1 + 2k_2 + \dots + nk_n = n,$$

is given by

$$\delta_{p^r,k}\left(\left\lfloor \frac{n-1}{p^r-1} \right\rfloor - \delta_{0,n \bmod p^r}\right),$$

where $\delta_{i,j}$ is the Kronecker delta and $\lfloor x \rfloor$ stands for the largest integer not exceeding x. This confirms a recent conjecture of Mircea Merca.

1. Introduction

The multinomial coefficients are defined by

$$\binom{k}{k_1, k_2, \dots, k_n} = \frac{k!}{k_1! k_2! \cdots k_n!},$$

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where $k = k_1 + k_2 + \cdots + k_n$. Fine [1, p. 87] gave a connection between multinomial coefficients and binomial coefficients:

$$\sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \binom{k}{k_1, k_2, \ldots, k_n} = \binom{n-1}{k-1}. \tag{1.1}$$

Let $M_m(n,k)$ be the number of multinomial coefficients such that

$$\binom{k}{k_1, k_2, \ldots, k_n} = m, \quad \text{and} \quad k_1 + 2k_2 + \cdots + nk_n = n.$$

For example, we have $M_6(10,3) = 4$, since

$$10 = 1 + 2 + 7 = 1 + 3 + 6 = 1 + 4 + 5 = 2 + 3 + 5.$$

It is easy to see that $M_1(n,k) = \delta_{0,n \bmod k}$. Recently, applying Fine's formula (1.1), Merca [2] obtained new upper bounds involving $M_m(n,k)$ for the number of partitions of n into k parts. He also proved that

$$M_2(n,k) = \delta_{2,k} \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad M_p(n,k) = \delta_{p,k} \left(\left\lfloor \frac{n-1}{p-1} \right\rfloor - \delta_{0,n \bmod p} \right),$$

where p is an odd prime.

In this paper, we shall prove the following result, which was conjectured by Merca [2, Conjecture 1].

Theorem 1. *Let p be a prime and let n, k, r be positive integers with $p^r > 2$. Then*

$$M_{p^r}(n,k) = \delta_{p^r,k} \left(\left\lfloor \frac{n-1}{p^r-1} \right\rfloor - \delta_{0,n \bmod p^r} \right).$$

Merca [2] pointed out that, when m is not a prime power, the formula for $M_m(n,k)$ is more involved. For example, we have

$$M_{10}(n,k) = \delta_{10,k} \left(\left\lfloor \frac{n-1}{9} \right\rfloor - \delta_{0,n \bmod 10} \right) + \delta_{5,k} \left(\left\lfloor \frac{n+1}{6} \right\rfloor - \delta_{0,n \bmod 5} - \delta_{0,n \bmod 6} \right).$$

2. Proof of Theorem 1

We need the following result.

Lemma 2. *Let n and k be two positive integers with $2 \leqslant k \leqslant \frac{n}{2}$. Then the binomial coefficient $\binom{n}{k}$ is not a prime power.*