Journal of Number Theory 147 (2015) 594–604

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On pairs of four prime squares and powers of two

Liqun Hu [a,b], Huafeng Liu [a,*]

[a] *Department of Mathematics, Shandong University, Jinan, Shandong 250100, PR China*
[b] *Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, PR China*

ARTICLE INFO

Article history:
Received 26 May 2014
Received in revised form 27 August 2014
Accepted 27 August 2014
Available online 6 October 2014
Communicated by David Goss

MSC:
11P32
11P05
11P55

Keywords:
Hardy–Littlewood method
Goldbach–Linnik problem
Powers of 2

ABSTRACT

It was proved that for $k = 584$, every pair of large positive even integers satisfying some necessary conditions can be represented in the form of a pair of four prime squares and k powers of 2. In this paper, we sharpen the value of k to 142.

1. Introduction

In 1951 and 1953, Linnik established the following “almost Goldbach” result that each large even integer N is a sum of two primes p_1, p_2 and a bounded number of powers of 2, namely

* Corresponding author.
E-mail addresses: huliqun@ncu.edu.cn (L. Hu), hfliu_sdu@hotmail.com (H. Liu).

http://dx.doi.org/10.1016/j.jnt.2014.08.006
0022-314X/

$$N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{1.1}$$

Later Gallagher [1] established a stronger result by a different method. An explicit value for the number k of powers of 2 was firstly established by the first author, Liu, Liu and Wang [9], who found that $k = 54\,000$ is acceptable. The original value for the number k was subsequently improved by Li [4], Wang [13] and Li [5]. In 2002, Heath-Brown and Puchta [2] applied a rather different approach to this problem and showed that $k = 13$ is acceptable. In 2003, Pintz and Ruzsa [12] proved $k = 7$ under GRH and announced that $k = 8$ is acceptable unconditionally.

Hua [3] proved that for each large integer $n \equiv 5 (\bmod 24)$ can be written as a sum of five squares of primes in 1938. In 1999, Liu, Liu and Zhan [10] showed that every large even integer n can be written as a sum of four squares of primes and a bounded number of powers of 2, namely

$$n = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}. \tag{1.2}$$

Subsequently Liu and Liu [8] got that $k = 8330$ suffices. Later Liu and Lü [11] improved the value of k of (1.2) to 165 and Li [6] improved it to 151.

Recently, Liu [7] first considered the result on simultaneous representation of pairs of positive even integers $N_2 \gg N_1 > N_2$ with $N_1 \equiv N_2 (\bmod 8)$, in the form

$$\begin{cases} N_1 = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}, \\ N_2 = p_5^2 + p_6^2 + p_7^2 + p_8^2 + 2^{v_1} + \cdots + 2^{v_k}, \end{cases} \tag{1.3}$$

where k is a positive integer. He proved that the simultaneous equations (1.3) are solvable for $k = 584$. In this short paper, we sharpen this result considerably by establishing the following theorem.

Theorem 1.1. *For $k = 142$, Eqs. (1.3) are solvable for every pair of sufficiently large positive even integers N_1 and N_2 satisfying $N_2 \gg N_1 > N_2$ and $N_1 \equiv N_2 (\bmod 8)$.*

We establish Theorem 1.1 by means of the Hardy–Littlewood method in combination with some new methods of Zhao [14].

Notation. Throughout this paper, the letter ε denotes a positive constant which is arbitrarily small but may not the same at different occurrences. p and v denote a prime number and a positive integer, respectively.

2. Outline of the method

Here we give an outline for the proof of Theorem 1.1.

In order to apply the Hardy–Littlewood method, we set