Journal of Number Theory 133 (2013) 3678–3702

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Ranks of partitions modulo 10

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A R T I C L E   I N F O

Article history:
Received 8 March 2013
Revised 8 May 2013
Accepted 9 May 2013
Available online 17 July 2013
Communicated by David Goss

Keywords:
Generating functions
Partitions
Rank
Theta identities
Ramanujan

A B S T R A C T

In 1954, A.O.L. Atkin and H.P.F. Swinnerton-Dyer established the generating functions for rank differences modulo 5 and 7 for partition functions. In this paper, we derive formulas for the generating functions of ranks of partitions modulo 10 and some inequalities between them.

1. Introduction

Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan discovered and later proved the following three famous congruences:

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.05.014

In order to find a combinatorial interpretation for Ramanujan's congruences, in 1944, F.J. Dyson [6] defined the rank of a partition to be the largest part minus the number of parts. If $N(m,n)$ is defined to be the number of partitions of n with rank m, then the generating function for $N(m,n)$ is given by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}. \tag{1.4}$$

Here and throughout, we use the notations

$$\begin{aligned}
(x_1,x_2,\dots,x_k;q)_m &:= \prod_{n=0}^{m-1}\bigl(1-x_1q^n\bigr)\bigl(1-x_2q^n\bigr)\cdots\bigl(1-x_kq^n\bigr),\\
(x_1,x_2,\dots,x_k;q)_\infty &:= \prod_{n=0}^{\infty}\bigl(1-x_1q^n\bigr)\bigl(1-x_2q^n\bigr)\cdots\bigl(1-x_kq^n\bigr),\\
[x_1,x_2,\dots,x_k;q]_\infty &:= (x_1,q/x_1,x_2,q/x_2,\dots,x_k,q/x_k;q)_\infty,\\
J_{a,b} &:= \bigl(q^a,q^{b-a},q^b;q^b\bigr),\\
\overline{J}_{a,b} &:= \bigl(-q^a,-q^{b-a},q^b;q^b\bigr),\\
J_b &:= \bigl(q^b;q^b\bigr)_\infty,\\
\overline{J}_b &:= \bigl(-q^b;q^b\bigr)_\infty,
\end{aligned}$$

and we require $|q|<1$ for absolute convergence.

Let $N(s,l,n)$ denote the number of partitions of n whose rank is congruent to s modulo l. Dyson then conjectured

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leqslant k \leqslant 4 \tag{1.5}$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{5}, \quad 0 \leqslant k \leqslant 6. \tag{1.6}$$

It is easy to see that the above two conjectures imply Ramanujan's congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, respectively. Dyson's conjectures were first proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [3] in 1954. In fact, they established the generating functions for every rank difference $N(s,l,ln+d)-N(t,l,ln+d)$ with $l=5$ or 7 and $0 \leqslant d,s,t \leqslant l$. Although Dyson's rank fails to explain Ramanujan's congruence (1.3) combinatorially, the method developed by Atkin and Swinnerton-Dyer [3] is widely used to get rank differences for other types of partitions ranks (see [16–18], for example). Besides (1.5) and (1.6), more relations between ranks of partitions modulo 5 and 7 have been obtained. As examples, we have