Journal of Number Theory 133 (2013) 3703–3716

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

An extensive analysis of the parity of broken 3-diamond partitions

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ARTICLE INFO

Article history:
Received 11 June 2012
Revised 30 April 2013
Accepted 20 May 2013
Available online 17 July 2013
Communicated by David Goss

MSC:
primary 11P83
secondary 05A17

Keywords:
Broken k-diamonds
Congruences
Modular forms
Partitions

ABSTRACT

In 2007, Andrews and Paule introduced the family of functions $\Delta_k(n)$ which enumerate the number of broken k-diamond partitions for a fixed positive integer k. Since then, numerous mathematicians have considered partitions congruences satisfied by $\Delta_k(n)$ for small values of k. In this work, we provide an extensive analysis of the parity of the function $\Delta_3(n)$, including a number of Ramanujan-like congruences modulo 2. This will be accomplished by **completely** characterizing the values of $\Delta_3(8n+r)$ modulo 2 for $r \in \{1,2,3,4,5,7\}$ and any value of $n \geqslant 0$. In contrast, we conjecture that, for any integers $0 \leqslant B < A$, $\Delta_3(8(An+B))$ and $\Delta_3(8(An+B)+6)$ is infinitely often even and infinitely often odd. In this sense, we generalize Subbarao's Conjecture for this function Δ_3. To the best of our knowledge, this is the first generalization of Subbarao's Conjecture in the literature.

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[1] S. Radu was supported by DK grant W1214-DK6 of the Austrian Science Funds FWF.
[2] J.A. Sellers gratefully acknowledges the support of the Austrian American Educational Commission which supported him during the Summer Semester 2012 as a Fulbright Fellow at the Johannes Kepler University, Linz, Austria.

http://dx.doi.org/10.1016/j.jnt.2013.05.009

1. Introduction

Broken k-diamond partitions were introduced in 2007 by Andrews and Paule [2]. These are constructed in such a way that the generating functions of their counting sequences $(\Delta_k(n))_{n\geqslant 0}$ are closely related to modular forms. Namely,

$$\sum_{n=0}^{\infty}\Delta_k(n)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{(2k+1)n})}{(1-q^n)^3(1-q^{(4k+2)n})}$$
$$=q^{(k+1)/12}\frac{\eta(2\tau)\eta((2k+1)\tau)}{\eta(\tau)^3\eta((4k+2)\tau)},\quad k\geqslant 1,$$

where we recall the Dedekind eta function

$$\eta(\tau):=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\quad (q=e^{2\pi i\tau}).$$

In their original work, Andrews and Paule proved that, for all $n\geqslant 0$,

$$\Delta_1(2n+1)\equiv 0\pmod 3. \tag{1.1}$$

They also conjectured a few other congruences modulo 2 satisfied by certain families of broken k-diamond partitions.

Since then, a number of authors have provided proofs of additional congruences satisfied by broken k-diamond partitions. Hirschhorn and Sellers [5] provided a new proof of (1.1) above as well as elementary proofs of the following parity results: For all $n\geqslant 0$,

$$\begin{aligned}\Delta_1(4n+2)&\equiv 0\pmod 2,\\ \Delta_1(4n+3)&\equiv 0\pmod 2,\\ \Delta_2(10n+2)&\equiv 0\pmod 2,\quad\text{and}\\ \Delta_2(10n+6)&\equiv 0\pmod 2\end{aligned}$$

The third result in the list above appeared in [2] as a conjecture while the other three did not. Soon after the publication of [5], Chan [3] provided a different proof of the parity results for Δ_2 mentioned above as well as a number of congruences modulo powers of 5. Subsequently, Paule and Radu [7] also proved a number of congruences modulo 5 for broken 2-diamond partitions, and they also shared conjectures related to broken 3-diamond partitions modulo 7 and broken 5-diamond partitions modulo 11. (Two of these conjectures have recently been proven by Xiong [12].)

Our goal in this work is to focus on parity results satisfied by $\Delta_3(n)$. The parity of this function has been studied, at least partially, by Radu and Sellers [10] who proved (among other things) that, for all $n\geqslant 0$,