Journal of Number Theory 133 (2013) 3739–3754

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Modular forms and effective Diophantine approximation

M. Ram Murty [1], Hector Pasten [*,2]

Department of Mathematics and Statistics, Queen's University, Jeffery Hall, University ave., Kingston, ON K7L 3N6, Canada

A R T I C L E I N F O

Article history:
Received 21 November 2012
Revised 21 May 2013
Accepted 25 May 2013
Available online 18 July 2013
Communicated by Michael A. Bennett

MSC:
primary 11F11, 11D75
secondary 11G05

Keywords:
Effective Diophantine approximation
Modular forms
Unit equation
ABC conjecture

A B S T R A C T

After the work of G. Frey, it is known that an appropriate bound for the Faltings height of elliptic curves in terms of the conductor (Frey's height conjecture) would give a version of the ABC conjecture. In this paper we prove a partial result towards Frey's height conjecture which applies to all elliptic curves over $\mathbb{Q}$, not only Frey curves. Our bound is completely effective and the technique is based in the theory of modular forms. As a consequence, we prove effective explicit bounds towards the ABC conjecture of similar strength to what can be obtained by linear forms in logarithms, without using the latter technique. The main application is a new effective proof of the finiteness of solutions to the S-unit equation (that is, S-integral points of $\mathbb{P}^1 - \{0, 1, \infty\}$), with a completely explicit and effective bound, without using any variant of Baker's theory or the Thue–Bombieri method.

Contents

* Corresponding author.
E-mail addresses: murty@mast.queensu.ca (M.R. Murty), hpasten@gmail.com (H. Pasten).

[1] The author has been partially funded by an NSERC Discovery grant.
[2] The author has been partially supported by an Ontario Graduate Scholarship.

0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.05.006

1. Introduction

A central problem in number theory is to establish the finiteness of integral or rational solutions to a Diophantine equation. The proof of such finiteness results often gives an upper bound for the number of solutions, while obtaining upper bounds for the *size* (or more precisely, *height*) of the solutions is a much harder problem. Results of the latter type are called *effective* since, in theory, a bound for the height of the solutions reduces the search for solutions to a finite amount of computation. For later reference, we denote the (logarithmic) *height* of a rational number $q \in \mathbb{Q}$ by

$$h(q) = \log \max\{|a|, |b|\}$$

where a, b are coprime integers with $q = a/b$.

Effective results are difficult to obtain, and essentially the only general approaches are Baker's theory of linear forms in logarithms along with the p-adic and elliptic analogues of it, and Bombieri's improvement of Thue's method [2].

The purpose of this note is to introduce another approach for obtaining effective finiteness results. The technique that we present is based on the theory of modular forms, and it originates in the known approaches to attack the ABC conjecture using elliptic curves and modular forms, which we discuss below. Using this approach we provide an 'algebro-geometric proof' of the following effective version of Mahler's theorem [13, p. 724] on the S-unit equation, a topic classically studied by means of analytic techniques.

Theorem 1.1. *Let S be a finite set of primes in $\mathbb{Z}$ and let P be the product of the elements of S. If $U, V \in \mathbb{Z}_S^\times$ satisfy $U + V = 1$ then*

$$\max\{h(U), h(V)\} < 4.8P \log P + 13P + 25.$$

Here, $\mathbb{Z}_S^\times$ denotes the group of units of the ring $\mathbb{Z}_S$ of rational S-integers. Moreover, as we vary the set S we get

$$\max\{h(U), h(V)\} < 4P \log P + O(P \log \log P).$$

The S-unit equation is a relevant case of finiteness result since several Diophantine problems can be reduced to it. Although this is not the first 'algebro-geometric' proof of finiteness of $\mathbb{Z}_S$-solutions to the unit equation (see the important work of M. Kim [11], where the result is stated in terms of $\mathbb{Z}_S$-points of $\mathbb{P}^1 - \{0, 1, \infty\}$ and it is attributed to Siegel), our method is effective and gives explicit constants.