Journal of Number Theory 133 (2013) 3771–3795

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

New extensions of some classical theorems in number theory

Magdalena Bănescu, Dumitru Popa *

Department of Mathematics, Ovidius University of Constanta, Bd. Mamaia 124, 900527 Constanta, Romania

ARTICLE INFO

Article history:
Received 12 February 2012
Revised 13 March 2013
Accepted 22 May 2013
Available online 19 July 2013
Communicated by David Goss

MSC:
primary 11N56
secondary 11N37

Keywords:
Arithmetic functions
Asymptotic results
Polya theorem

ABSTRACT

In this paper we show that for $k \in \mathbb{N} \cup \{0\}$, under natural assumptions on the functions g and h, for a large class of Riemann integrable functions $f : [0,1]^{k+1} \to \mathbb{R}$ (not all, for $k \in \mathbb{N}$; and all, for $k = 0$), the following equality holds

$$\lim_{x\to\infty} \frac{1}{h(x)} \sum_{n \leqslant x} f\left(\frac{n}{x}, \frac{\ln_1 n}{\ln_1 x}, \ldots, \frac{\ln_k n}{\ln_k x}\right) g(n)$$

$$= \int_0^1 f(x, \underbrace{1, \ldots, 1}_{k\text{-times}})\, dx.$$

Using these results for prime numbers, we obtain some new extensions of the classical version from 1917 Polya's theorem in number theory.

1. Introduction

In the famous paper published in 1896, see [7], J. Hadamard showed that if $\alpha > 1$

$$\lim_{x\to\infty} \frac{1}{x} \sum_{p \leqslant x,\, p \text{ prime}} \left(\ln \frac{x}{p}\right)^{\alpha-1} \ln p = \Gamma(\alpha) \tag{H}$$

* Corresponding author.
E-mail addresses: mbanescu@yahoo.com (M. Bănescu), dpopa@univ-ovidius.ro (D. Popa).

0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.05.011

where Γ is the Euler gamma function and then he used this result in order to prove the prime number theorem.

Few years later, Landau in 1900, see [10], using the prime number theorem, showed that Hadamard's formula (H) is true.

In a classical paper from 1917, see [14], based on the prime number theorem, Polya showed that if $f:[0,1]\to\mathbb{R}$ is a Riemann integrable function on $[0,1]$, then

$$\lim_{x\to\infty}\frac{\ln x}{x}\sum_{p\leqslant x,\, p \text{ prime}} f\left(\frac{p}{x}\right)=\int_0^1 f(x)\,dx.$$

Some recent applications of Polya's theorem can be found in [4].

In 1977, see [15], Radoux showed that

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\varphi(k)=\frac{6}{\pi^2}\int_0^1 xf(x)\,dx,$$

for all functions $f:[0,1]\to\mathbb{R}$ such that $xf(x)$ is continuous on $[0,1]$, φ is Euler's totient function, see also [8] and [13].

In the present paper we will show that all these apparently different results are in fact of the same kind, see Theorem 2, Theorem 3, Theorem 4, Corollary 3 and Theorem 5. Our approach is the following: first, we prove the results for polynomials, then for continuous functions and at the end for Riemann integrable functions.

Let us fix first some notations and notions.

Let $a\in\mathbb{R}\cup\{-\infty\}$, $f:(a,\infty)\to\mathbb{R}$ and $g:(a,\infty)\to\mathbb{R}$ with the property that there exists $b\geqslant a$ with $g(x)\neq 0$ for all $x\in(b,\infty)$. Throughout the paper, we will use the following notation: $f(x)\sim g(x)$ as $x\to\infty$ if and only if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$.

We recall that if $g:\mathbb{N}\to[0,\infty)$ is a function, its summatory function $G:(0,\infty)\to[0,\infty)$ is defined by $G(x)=\sum_{n\leqslant x}g(n)$, see [2, page 39].

If $h:(0,\infty)\to\mathbb{R}$ is such that there exists $x_0>0$ with $h(x)\neq 0$ for all $x\geqslant x_0$ and $g:\mathbb{N}\to[0,\infty)$, we say that the summatory function of g is equivalent to h if and only if $\sum_{n\leqslant x}g(n)\backsim h(x)$ as $x\to\infty$.

We denote by e the Euler number and we define the sequence $(e_k)_{k\geqslant 0}$ by $e_0=1$, $e_{k+1}=e^{e_k}$ for $k\geqslant 0$. We also define $\ln_1 x=\ln x=\log_e x$ for $x>0$ and $\ln_{k+1}x=\ln(\ln_k x)$ for $k\geqslant 1$ and $x>e_{k-1}$.

Let k be a natural number. We write

$$C\left([0,1]^k\right):=\left\{f\colon\ [0,1]^k\to\mathbb{R}\ \middle|\ f \text{ continuous on } [0,1]^k\right\},$$

which is a real linear space with respect to usual addition and scalar multiplication for functions and a Banach space with respect to the uniform norm i.e. $\|f\|_u=\sup_{(x_1,\dots,x_k)\in[0,1]^k}|f(x_1,\dots,x_k)|$.