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Bounding differences in Jager Pairs

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ABSTRACT

Symmetrical subdivisions in the space of Jager Pairs for continued fractions-like expansions will provide us with bounds on their differences. Results will also apply to the classical regular and backwards continued fractions expansions, which are realized as special cases.

1. Introduction

Given a real number r and a rational number, written as the unique quotient $\frac{p}{q}$ of the relatively prime integers p and $q > 0$, our fundamental object of interest from diophantine approximation is the *approximation coefficient* $\theta(r, \frac{p}{q}) := q^2|r - \frac{p}{q}|$. Small approximation coefficients suggest high quality approximations, combining accuracy with simplicity. For instance, the error in approximating π using $\frac{355}{113} = 3.1415920353982$ is smaller than the error of its decimal expansion to the fifth digit $3.14159 = \frac{314\,159}{100\,000}$. Since the former rational also has a much smaller denominator, it is of far greater quality than the latter. Indeed $\theta(\pi, \frac{355}{113}) < 0.0341$ whereas $\theta(\pi, \frac{314\,159}{100\,000}) > 26\,535$.

Since adding integers to fractions does not change their denominators, we have $\theta(r, \frac{p}{q}) = \theta(r - \lfloor r \rfloor, \frac{p}{q} - \lfloor r \rfloor)$, where $\lfloor r \rfloor$ is the largest integer smaller than or equal to r (a.k.a. the floor of r), allowing us to restrict our attention to the unit interval. Expanding an irrational initial seed $x_0 \in (0, 1) - \mathbb{Q}$ as an infinite regular continued fraction

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$$x_0 = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

provides us with the unique symbolic representation via the sequence $\{b_n\}_1^\infty$ of positive integers, known as the partial quotients or *digits of expansion* of x_0. For all $n \geqslant 0$, we label the approximation coefficient associated with the *convergent*

$$\frac{p_0}{q_0} := \frac{0}{1}, \qquad \frac{p_n}{q_n} := \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\cdots + \cfrac{1}{b_n}}}}, \quad n \geqslant 1$$

by θ_n and refer to the sequence $\{\theta_n\}_1^\infty$ as the *sequence of approximation coefficients*.

Much work has been done with this sequence, from its inception in the classical era to more recent excursion which establishes interesting connections with ergodic theory and hyperbolic geometry. For a concise description of some classical results concerning this sequence, refer to the introduction section of [2]. A more thorough treatment can be found in [3, Chapter 5]. Another well-known continued fraction theory is the *backward continued fraction* expansion:

$$x_0 = 1 - \cfrac{1}{b_1 + 1 - \cfrac{1}{b_2 + 1 - \cfrac{1}{b_3 + \cdots}}}$$

leading to a new unique sequence of digits, hence new sequences of convergents and approximation coefficients.

The focus of this paper is the *space of Jager Pairs*

$$\Gamma(x_0) := \big\{\big(\theta_{n-1}(x_0), \theta_n(x_0)\big),\ x_0 \in (0,1) - \mathbb{Q},\ n \geqslant 1\big\}.$$

The spaces corresponding to the regular and backwards continued fraction expansions were initially introduced and studied in [7] and [5] respectively. We are going to reveal an elegant symmetrical internal structure for both these spaces. Our approach is to treat the regular and continued fraction expansions as limiting cases for the two families of one parameter continued fraction-like expansions, first introduced by Haas and Molnar. Using simple plane geometry, we will provide in Corollary 4.4 upper and lower bounds for the growth rate of the associated sequence of approximation coefficients. For instance, knowing a priori that $b_2 = b_3 = 1$ and $b_4 = 3$ are the digits of the classical regular continued fraction expansion will allow us to obtain the bounds $|\theta_2 - \theta_1| < \frac{\sqrt{2}}{3}$ and $\frac{2\sqrt{2}}{7} < |\theta_3 - \theta_2| < \frac{3\sqrt{2}}{5}$.

2. Preliminaries

This section is a paraphrased summary of excerpts from [4,5], given for sake of completeness. In general, the fractional part of Möbius transformation which maps $[0,1]$ onto $[0,\infty]$ leads to expansion of real numbers as continued fractions. To characterize all these transformations, we recall that Möbius transformations are uniquely determined by their values for three distinct points. Thus, we will need to introduce a parameter for the image of an additional point besides 0 and 1, which we will naturally take to be ∞. Since our maps fix the real line, the image of ∞, denoted by $-k$, can take any value within the set of negative real numbers. We let $m \in \{0,1\}$ equal zero for orientation