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Threefield identities and simultaneous representations of primes by binary quadratic forms

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ABSTRACT

Kaplansky [2003] proved a theorem on the simultaneous representation of a prime p by two different principal binary quadratic forms. Later, Brink found five more like theorems and claimed that there were no others. By putting Kaplansky-like theorems into the context of threefield identities after Andrews, Dyson, and Hickerson, we find that there are at least two similar results not on Brink's list. We also show how such theorems are related to results of Muskat on binary quadratic forms.

1. Notation

Let q be a complex number with $0 < |q| < 1$. We recall some basic facts:

$$(x)_\infty = (x;q)_\infty := \prod_{i\geqslant 0}(1-q^i x), \tag{1.1}$$

and

$$j(x;q) := (x)_\infty (q/x)_\infty (q)_\infty = \sum_{n\in\mathbb{Z}}(-1)^n q^{\binom{n}{2}} x^n, \tag{1.2}$$

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where in the last line the equivalence of product and sum follows from Jacobi's triple product identity. Let a and m be integers with m positive. Define

$$J_{a,m} := j(q^a; q^m), \qquad \overline{J}_{a,m} := j(-q^a; q^m), \quad \text{and}$$
$$J_m := J_{m,3m} = \prod_{i\geqslant 1}(1-q^{mi}). \tag{1.3}$$

2. Introduction

Let Δ be a negative integer with $\Delta \equiv 0 \pmod 4$ (resp. $\Delta \equiv 1 \pmod 4$). Recall that the principal binary quadratic form $F(x,y)$ of discriminant Δ is defined to be $x^2 - \frac{\Delta}{4}y^2$ (resp. $x^2+xy+\frac{1-\Delta}{4}y^2$). Kaplansky [11] proved the following theorem on the simultaneous representation of a prime p by two different principal binary quadratic forms:

Theorem 2.1. *(See [11].) A prime p, where $p \equiv 1 \pmod{16}$, is representable by both or none of the quadratic forms x^2+32y^2 and x^2+64y^2. A prime p, where $p \equiv 9 \pmod{16}$, is representable by exactly one of the quadratic forms.*

Kaplansky proved his theorem using two well-known results: 2 *is a* 4*th power modulo a prime p if and only if p is represented by x^2+64y^2 (Gauss* [6, p. 530]) and -4 *is an* 8*th power modulo a prime p if and only if p is represented by x^2+32y^2 (Barrucand and Cohn* [2]). Using class field theory, Brink [3] was able to prove five more theorems similar to that of Kaplansky. Two of which are

Theorem 2.2. *(See [3, Theorem 1].) A prime $p \equiv 1 \pmod{20}$ is representable by both or none of x^2+20y^2 and x^2+100y^2, whereas a prime $p \equiv 9 \pmod{20}$ is representable by exactly one of these forms.*

Theorem 2.3. *(See [3, Theorem 4].) A prime $p \equiv 1, 65, 81 \pmod{112}$ is representable by both or none of x^2+14y^2 and x^2+448y^2, whereas a prime $p \equiv 9, 25, 57 \pmod{112}$ is representable by exactly one of these forms.*

In [3], Brink claims that these are the only results of their kind and gives a heuristic argument as support. As an example, Brink shows that there is no similar result for primes represented by x^2+128y^2 and x^2+256y^2. In [4], Brink gives elementary proofs of Theorems 2.1 and 2.2 and also shows that Theorem 2.1 is equivalent to a result of Glaisher [7]: *Let p be an odd prime and let h and h' be the class numbers corresponding to the discriminants $-4p$ and $-8p$ respectively. If $p \equiv 1 \pmod{16}$, then either both or none of h and h' are divisible by* 8; *if $p \equiv 9 \pmod{16}$, then exactly one of these class numbers is divisible by* 8.

It turns out that there are at least two pairs of discriminants for Kaplansky-like theorems on principal binary quadratic forms that are not on Brink's list. Our two new results read