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Computations of vector-valued Siegel modular forms ☆,☆☆

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ABSTRACT

We carry out some computations of vector-valued Siegel modular forms of degree two, weight $(k, 2)$ and level one, and highlight three experimental results: (1) we identify a rational eigenform in a three-dimensional space of cusp forms; (2) we observe that non-cuspidal eigenforms of level one are not always rational; (3) we verify a number of cases of conjectures about congruences between classical modular forms and Siegel modular forms. Our approach is based on Satoh's description of the module of vector-valued Siegel modular forms of weight $(k, 2)$ and an explicit description of the Hecke action on Fourier expansions.

1. Introduction

Computations of modular forms in general and Siegel modular forms in particular are of great current interest. Recent computations of Siegel modular forms on the paramodular group by Poor and Yuen [21] have led to the careful formulation by Brumer and

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Kramer [3] of the Paramodular Conjecture, a natural generalization of the Taniyama–Shimura–Weil Conjecture. Historically, computations of scalar-valued Siegel modular forms in the 1970s by Kurokawa [15] led to the discovery of the Saito–Kurokawa lift, a construction whose generalizations are still studied. In the 1990s, computations by Skoruppa [28] revealed some striking properties that some scalar-valued Siegel modular forms possess (namely, there are rational eigenforms of weights 24 and 26 in level 1 that span a two-dimensional space of cusp forms). These properties have yet to be explained. This paper is in the same spirit.

We carry out the first systematic computations of spaces of vector-valued Siegel modular forms of degree two and of weight $(k, 2)$. We do this in Sage [29] using a package co-authored by the second author, Raum, Skoruppa and Tornaría [22]. We then decompose these spaces using the action of the Hecke operators and check that the eigenforms we compute satisfy the Ramanujan–Petersson bound (see Proposition 3.1). We also observe two new phenomena: the existence of non-cuspidal eigenforms that are not defined over $\mathbb{Q}$ (see Proposition 3.3), and the existence of a three-dimensional space of cusp forms of level one that is reducible as a Hecke module over $\mathbb{Q}$ (see Proposition 3.2).

In addition, we verify two interesting conjectures on congruences: the first, due to Harder, has been previously verified in some cases by Faber and van der Geer [31]; the other, due to Bergström, Faber, van der Geer and Harder, has been previously verified in some cases by Dummigan [7]. Our approach to verifying these conjectures is to compute Hecke eigenvalues of Siegel modular forms in as direct a manner as possible, using Satoh's concrete description of Siegel modular forms of weight $(k, 2)$. This is a very different approach than the one taken by Faber and van der Geer and we verify cases that they do not (and vice versa). After determining a basis of eigenforms for the space, we use explicit formulas for the Hecke action on Fourier expansions to extract the Hecke eigenvalues. Our main results in this direction are Theorems 4.3 and 4.8, which summarize the cases of the two conjectures that we have verified.

Our computation of the spaces of Siegel modular forms uses the framework developed in [23], to which we refer the interested reader for a detailed description. The current paper focuses on those aspects relevant to vector-valued forms of weight $(k, 2)$ and on the consequences of these computations. We have made the data gathered in the process publicly available at [25] and at [19].

2. Vector-valued Siegel modular forms of weight $(k, 2)$

We recall the definition of a Siegel modular form of degree two. We consider the *full Siegel modular group* $\Gamma^{(2)}$ given by

$$\Gamma^{(2)} := \mathrm{Sp}(4, \mathbb{Z}) = \left\{ M \in \mathrm{M}(4, \mathbb{Z}) \colon {}^tM \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} M = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix} \right\}. \tag{1}$$