Journal of Number Theory 137 (2014) 67–92

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Equidistribution of generalized Dedekind sums and exponential sums ☆

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ARTICLE INFO

Article history:
Received 12 July 2013
Received in revised form 16 October 2013
Accepted 16 October 2013
Available online 28 December 2013
Communicated by David Goss

Keywords:
Dedekind sums
Exponential sums
Equidistribution

ABSTRACT

For the generalized Dedekind sums $s_{ij}(p,q)$ defined in association with the $x^i y^j$-coefficient of the Todd power series of the lattice cone in $\mathbb{R}^2$ generated by $(1,0)$ and (p,q), we associate an exponential sum. We obtain this exponential sum using the cocycle property of the Todd series of 2d cones and the nonsingular cone decomposition along with the continued fraction of q/p. Its Weil bound is given for the modulus q applying the purity theorem of the cohomology of the related $\bar{\mathbb{Q}}_\ell$-sheaf due to Denef and Loeser. The Weil type bound of Denef and Loeser fulfills the Weyl equidistribution criterion for $R(i,j)q^{i+j-2}s_{ij}(p,q)$. As a special case, we recover the equidistribution result of the classical Dedekind sums multiplied by 12 not using the modular weight of the Dedekind $\eta(\tau)$.

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☆ The work of B. Jun is supported by the National Research Foundation (NRF) grant funded by the Korea government (MEST) (NRF-2012R1A1A2007726). The work of J. Lee is supported by the National Research Foundation (NRF) grant funded by the Korea government (MEST) (NRF-2011-0023688).

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0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.10.020

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1. Introduction

Classical Dedekind sums $s(p,q)$ are defined for relatively prime integers p,q by

$$s(p,q)=\sum_{k=1}^{q}\left(\left(\frac{k}{q}\right)\right)\left(\left(\frac{kp}{q}\right)\right)$$

where $((x))$ denotes the value of the 1st periodic Bernoulli function at x:

$$((x))=\bar{B}_1(x):=\begin{cases} x-[x]-\frac{1}{2} & \text{for } x\notin\mathbb{Z},\\ 0 & \text{for } x\in\mathbb{Z}.\end{cases}$$

This appears important in describing the change of the Dedekind eta function

$$\eta(\tau)=e^{2\pi i\tau/24}\prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right),\quad \tau\in\mathfrak{h},$$

under modular transformations. $\eta(\tau)$ is a 24th root of the modular discriminant

$$\Delta(\tau)=(12\pi)^{12}\eta^{24}(\tau)$$

up to some constant.

Due to the modularity after 24th power, under modular transformation $\tau\mapsto A\tau$, for $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, its logarithm satisfies the following formula due to Dedekind:

$$\log\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\log\eta(\tau)+\frac{1}{4}\log\left(-(c\tau+d)^2\right)+\pi i\phi(A) \tag{1}$$