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Remarks on Euclidean minima ☆

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ABSTRACT

The Euclidean minimum $M(K)$ of a number field K is an important numerical invariant that indicates whether K is norm-Euclidean. When K is a non-CM field of unit rank 2 or higher, Cerri showed $M(K)$, as the supremum in the Euclidean spectrum $\mathrm{Spec}(K)$, is isolated and attained and can be computed in finite time. We extend Cerri's works by applying recent dynamical results of Lindenstrauss and Wang. In particular, the following facts are proved:

(1) For any number field K of unit rank 3 or higher, $M(K)$ is isolated and attained and Cerri's algorithm computes $M(K)$ in finite time.
(2) If K is a non-CM field of unit rank 2 or higher, then the computational complexity of $M(K)$ is bounded in terms of the degree, discriminant and regulator of K.

Contents

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1. Introduction

1.1. Background

A number field K is said to be norm-Euclidean if its ring of integers $\mathcal{O}_K$ is a Euclidean domain with respect to the algebraic norm $|N_K(\cdot)|$, that is, for all $x, y \in \mathcal{O}_K$, there exists $a \in \mathcal{O}_K$ such that $|N_K(x - ay)| < |N_K(y)|$. The Euclidean minimum of K is a numerical indicator of whether K is norm-Euclidean or not.

Definition 1.1. The *Euclidean minimum* of an element $x \in K$ is $m_K(x) = \inf_{y \in x + \mathcal{O}_K} |N_K(y)|$.

The *Euclidean spectrum* of the number field K is the image $\{m_K(x)\colon x \in K\}$ and the *Euclidean minimum* of K is $M(K) = \sup_{x \in K} m_K(x)$.

It is known that K is norm-Euclidean if $M(K) > 1$ and is not norm-Euclidean if $M(K) < 1$. When $M(K) = 1$, it was proved by Cerri [Cer06] that if the unit rank of K is at least 2 then it is not norm-Euclidean.

One can easily check that $m_K(x) \geqslant 0$ and $M(K) > 0$. When K is totally real it is part of a conjecture of Minkowski that $M(K) \leqslant 2^{-d}\sqrt{D_K}$ where d and D_K denote respectively the degree and discriminant of K. The conjecture has been proved only for number fields of low degrees. However, weaker general upper bounds are available: for totally real fields Chebotarev proved $M(K) \leqslant 2^{-\frac{d}{2}}\sqrt{D_K}$ (see for example [HW79, §24.9]). For general number fields (not necessarily totally real), Bayer Fluckiger showed in [BF06] that

$$M(K) \leqslant 2^{-d} D_K. \tag{1.1}$$

In the rest of this paper, we will always write

$$\overline{K} = K \otimes_{\mathbb{Q}} \mathbb{R}. \tag{1.2}$$