Journal of Number Theory 137 (2014) 122–129

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Fundamental units of real quadratic fields of odd class number ☆

Zhe Zhang [a,*], Qin Yue [b,c]

[a] School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, PR China
[b] Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, PR China
[c] SKL of mathematical engineering and advanced computing, PR China

ARTICLE INFO

Article history:
Received 24 July 2013
Received in revised form 5 August 2013
Accepted 7 October 2013
Available online 31 December 2013
Communicated by David Goss

MSC:
11R27
11D09

Keywords:
Real quadratic fields
Fundamental units
Diophantine equations

ABSTRACT

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field with odd class number and its fundamental unit $\epsilon_d = x + y\sqrt{d} > 1$ satisfies $N_{K/\mathbb{Q}}(\epsilon_d) = 1$. We give some congruence relations about x, y explicitly.

1. Introduction

Throughout this paper, let d be a squarefree positive integer and $K = \mathbb{Q}(\sqrt{d})$ a real quadratic field. Let $\epsilon_d = x + y\sqrt{d} > 1$ be the fundamental unit of K with x, y positive rational numbers. It is well-known that K has odd class number with $N_{K/\mathbb{Q}}(\epsilon_d) = 1$ if

☆ Research partially supported by National Key Basic Research Program of China (Grant No. 2013CB834202) and National Natural Science Foundation of China (Grant Nos. 11171150 and 11171317).
* Corresponding author.
E-mail addresses: lmlz@mail.ustc.edu.cn (Z. Zhang), yueqin@nuaa.edu.cn (Q. Yue).

0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2013.10.019

and only if $d = p, 2p$ or p_1p_2 with $p \equiv p_1 \equiv p_2 \equiv 3 \bmod 4$ primes (cf. [2, p. 163]). If $d = p, 2p$ with $p \equiv 3 \bmod 4$ or $d = p_1p_2$ with $p_1p_2 \equiv 1 \bmod 8$, then x, y are integers. If $d = p_1p_2$ with $p_1p_2 \equiv 5 \bmod 8$, it can happen that x, y are not integers, if so, then $\epsilon^3_{p_1p_2}$ does have integral coefficients. In order to avoid fractions, we will temporarily let $\epsilon_d = x + y\sqrt{d}$, where the positive integer pair (x, y) is the fundamental integer solution of the Diophantine equation

$$x^2 - dy^2 = 1, \tag{1.1}$$

and we shall refer to ϵ_d as the *fundamental integral unit* of $K = \mathbb{Q}(\sqrt{d}\,)$ (cf. [1]). Thus when $d = p, 2p$ with $p \equiv 3 \bmod 4$ or $d = p_1p_2$ with $p_1p_2 \equiv 1 \bmod 8$, the fundamental unit of K is the fundamental integral unit. And when $d = p_1p_2 \equiv 5 \bmod 8$, if the fundamental unit of K is not the fundamental integral unit, then its third power is the fundamental integral unit of K.

The aim of this paper is to prove the following theorem.

Theorem 1.1. *Let $K = \mathbb{Q}(\sqrt{d}\,)$ be a real quadratic field with odd class number and let $\epsilon_d = x + y\sqrt{d} > 1$ be the fundamental integral unit of K, then we have*

(1) *If $d = p$ with $p \equiv 3 \bmod 4$, then $x \equiv 0 \bmod 2$. More precisely, if $p \equiv 3 \bmod 8$, then $x \equiv 2 \bmod 4$; if $p \equiv 7 \bmod 8$, then $x \equiv 0 \bmod 4$.*
(2) *If $d = 2p$ with $p \equiv 3 \bmod 4$, then $y \equiv 0 \bmod 2$ and $x + y \equiv 3 \bmod 4$.*
(3) *If $d = p_1p_2$ with $p_1 \equiv p_2 \equiv 3 \bmod 4$, then $x \equiv 3 \bmod 4$, $y \equiv 0 \bmod 4$.*

The proof of Theorem 1.1 is given in Section 3. The main idea is to show that $K(\sqrt{\epsilon_d}\,)/K$ is ramified at the dyadic prime ideal of K (i.e., the prime ideal of K lying above 2), which would lead to a contradiction if Theorem 1.1 was false. Before we prove our theorem, we give some useful properties of 2-adic local fields in Section 2. In the last section, we give some applications of Theorem 1.1 in solving Diophantine equations.

2. Local computation

In this section, we compile several results for proving Theorem 1.1. For local field F, we let $\mathcal{O}_F$ be the ring of integers of F and U_F the unit group of $\mathcal{O}_F$. Let $U_F^{(n)} = 1 + \pi^n\mathcal{O}_F$ where π is a uniformizer of F.

Lemma 2.1. *Suppose $F = \mathbb{Q}_2(\sqrt{-1}\,)$. Then $\pi = -1 + \sqrt{-1}$ is a uniformizer of F and*

(1) *$U_F^{(5)} = (U_F^{(3)})^2$, i.e., every element of $U_F^{(5)}$ is a square.*
(2) *$F(\sqrt{3}\,) = F(\sqrt{-3}\,)$ is unramified over F.*

Proof. (1) That π is a uniformizer is because it is a root of the Eisenstein polynomial $x^2 + 2x + 2$. Since $F/\mathbb{Q}_2$ is totally ramified, $U_F = U_F^{(1)}$ and $[U_F : U_F^{(5)}] = 16$. In our