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On the coefficients of an eigenform

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ABSTRACT

Lehmer's conjecture on Ramanujan's tau function $\tau(n)$ asserts that $\tau(n) \neq 0$ for any $n \geqslant 1$. We consider a variant of Lehmer's conjecture. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a normalized eigenform, where the a_n are rational integers for all n, and let $g(n)$ be a polynomial with integer coefficients. We bound the number of $n \leqslant x$ such that a_n and $g(n)$ have no common factor.

1. Introduction

Ramanujan's tau function is denoted by

$$\Delta(z) = \sum_{n=0}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} \left(1 - q^n\right)^{24} = q - 24q^2 + 252q^3 + \cdots,$$

where $q = e^{2\pi i z}$. The well known Lehmer's conjecture [5,6] asserts that

$$\tau(n) \neq 0$$

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for any $n \geqslant 1$. This conjecture remains open for a long time and seems quite difficult. More generally, one can ask the vanishing of the Fourier coefficients of a Hecke eigenform.

In a recent work by V.K. Murty [8], a variant of Lehmer's conjecture has been considered. Let $S_k(\Gamma_0(N))$ be the vector space of cusp forms of weight k for $\Gamma_0(N)$, and

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k\bigl(\Gamma_0(N)\bigr) \cap \mathbb{Z}[\![q]\!]$$

be a normalized eigenform, where the coefficients a_n are rational integers. V.K. Murty's problem is that whether it is true that

$$\sharp\bigl\{n \leqslant x\colon\ (n, a_n) = 1\bigr\} = o(x)$$

as $x \to \infty$. As explained in [8], this is a variant of Lehmer's conjecture. Moreover, V.K. Murty showed that

$$\sharp\bigl\{n \leqslant x\colon\ (n, a_n) = 1\bigr\} \ll \frac{x}{\log\log\log x}.$$

This means that for almost all n, a_n and n have common divisor > 1.

Note that for the Euler ϕ-function, Erdős [1] proved in 1948 that

$$\sharp\bigl\{n \leqslant x\colon\ \bigl(n, \phi(n)\bigr)\bigr\} \sim \frac{e^{-\gamma} x}{\log\log\log x}.$$

Therefore V.K. Murty's result can be seen as a modular analogue of the result of Erdős. We remark that Murty's proof is based on the techniques of Erdős and those of [7,9].

When $f(z)$ has complex multiplication with weight $k = 2$, a stronger result was obtained by Gun and V.K. Murty [2]. They proved that there is a constant U_f such that

$$\sharp\bigl\{n \leqslant x\colon\ (n, a_n) = 1\bigr\} = \bigl(1 + o(1)\bigr)\frac{U_f x}{(\log x \log\log\log x)^{\frac{1}{2}}}.$$

An analogous result was established by Laptyeva [4] for $k \geqslant 4$ under certain assumption.

In this note we will establish a general theorem analogous V.K. Murty's result. Our approach is more simpler, and based on the combinatorial sieve and a result of Serre.

Theorem 1. *Let $N \geqslant 1$ be an integer and $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N)) \cap \mathbb{Z}[\![q]\!]$ be a normalized eigenform, where the a_n are rational integers. Moreover, let $g(n)$ be a polynomial with integer coefficients and degree d. Suppose that $\rho(p)$ is the number of the solutions of $g(n) \equiv 0 \pmod p$ and satisfies*

$$\rho(p) < p$$