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Hodge theory and the Mordell–Weil rank of elliptic curves over extensions of function fields

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A R T I C L E I N F O

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We use Hodge theory to prove a new upper bound on the ranks of Mordell–Weil groups for elliptic curves over function fields after regular geometrically Galois extensions of the base field, improving on previous results of Silverman and Ellenberg, when the base field has characteristic zero and the supports of the conductor of the elliptic curve and of the ramification divisor of the extension are disjoint.

1. Introduction

For every field F let $\overline{F}$ denote its separable closure. For every field extension $K|F$ and variety Z defined over F let Z_K denote the base change of Z to K. Let $\mathcal{C}$ be a smooth projective geometrically irreducible curve defined over a field k and let F, g denote its function field and its genus, respectively. Let $\pi : \mathcal{C}' \to \mathcal{C}$ be a finite regular geometrically Galois cover defined over k. Let F' be the function field of $\mathcal{C}'$ and let S be

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the reduced ramification divisor of the cover π. Let $G = \mathrm{Aut}(\mathcal{C}'_{\bar{k}}|\mathcal{C}_{\bar{k}})$ and let Σ be the image of $\mathrm{Gal}(\bar{k}|k)$ in $\mathrm{Aut}(G)$ with respect to the natural action of $\mathrm{Gal}(\bar{k}|k)$ on G.

Next we recall the definition of Ellenberg's constant. Let V be the real vector space spanned by the irreducible complex-valued characters of $G \rtimes \Sigma$, and let W be the real vector space spanned by the irreducible complex-valued characters of G. We say that a vector v in V (resp. in W) is non-negative if its inner product with each irreducible representation of $G \rtimes \Sigma$ (resp. of G) is non-negative. Let $c \in V$ be the coset character of $G \rtimes \Sigma$ attached to Σ, and let $r \in W$ be the regular character of G. Ellenberg defines the constant $\epsilon(G, \Sigma)$ as the maximum of the inner product $\langle v, c\rangle$ over all $v \in V$ such that

(i) v is non-negative;
(ii) $r - R(v)$ is non-negative, where $R : V \to W$ is the restriction map.

The region of V defined by these two conditions above is a compact polytope, so $\epsilon(G, \Sigma)$ is well-defined.

Let E be a non-isotrivial elliptic curve over F. By the Lang–Néron theorem (see [1]) the group $E(F')$ is finitely generated. Let c_E denote the degree of the conductor of E and let d_E denote the degree of the minimal discriminant of E. Our main result is the following

Theorem 1.1. *Assume that k has characteristic zero, and the supports of S and of the conductor of E are disjoint. Then*

$$\mathrm{rank}\big(E\big(F'\big)\big) \leqslant \epsilon(G, \Sigma)\big(c_E - d_E/6 + 2g - 2 + \deg(S)\big). \tag{1.1.1}$$

Let $O(G, \Sigma)$ be the cardinality of the set of orbits of G with respect to the action of Σ. It is easy to prove that $O(G, \Sigma) = \epsilon(G, \Sigma)$ when G is an abelian group (see Proposition 2.11 of [3]). Hence we have the following immediate

Corollary 1.2. *Assume that k has characteristic zero, the supports of S and of the conductor of E are disjoint, and G is abelian. Then*

$$\mathrm{rank}\big(E\big(F'\big)\big) \leqslant O(G, \Sigma)\big(c_E - d_E/6 + 2g - 2 + \deg(S)\big). \tag{1.2.1}$$

The upper bound:

$$\mathrm{rank}\big(E\big(F'\big)\big) \leqslant O(G, \Sigma)(c_E + 4g - 4), \tag{1.2.2}$$

was first proved by Silverman in [8] under the assumption that G is abelian, the cover π is unramified (i.e. S is empty), k is a number field, and a weak form of Tate's conjecture holds for $\mathcal{E}'$ where $g' : \mathcal{E}' \to \mathcal{C}'$ is the unique relatively minimal elliptic surface over