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New convolution identities for hypergeometric Bernoulli polynomials

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A B S T R A C T

New convolution identities of hypergeometric Bernoulli polynomials are presented. Two different approaches to proving these identities are discussed, corresponding to the two equivalent definitions of hypergeometric Bernoulli polynomials as Appell sequences.

1. Introduction

It is well known that the Euler–Maclaurin Summation (EMS) formula given by

$$\sum_{k=0}^{n} f(k) = \int_{0}^{n} f(x)\,dx + \frac{1}{2}\left[f(n) + f(0)\right] + \sum_{k=2}^{\infty} \frac{B_k}{k!}\left[f^{(k-1)}(n) - f^{(k-1)}(0)\right] \qquad (1)$$

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is extremely useful for approximating sums and integrals and for deriving special formulas. Here, B_n are the Bernoulli numbers defined by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

For example, if we set $f(x) = x^p$ in (1) and use the fact that $B_0 = 1$ and $B_1 = -1/2$, then we obtain the classical sums of powers formula first discovered by Jacob Bernoulli:

$$\sum_{k=1}^{n} k^p = n^p + \sum_{k=0}^{p} \frac{p!}{k!(p-k+1)!} B_k n^{p+1-k}$$

Consider next the special case of the EMS formula where $n = 1$, which we shall write in the form

$$\int_0^1 f(x)\,dx = \frac{1}{2}\left[f(1) + f(0)\right] - \sum_{k=2}^{\infty} \frac{B_k}{k!}\left[f^{(k-1)}(1) - f^{(k-1)}(0)\right] \tag{2}$$

If we again set $f(x) = x^n$ in (2), then we obtain the classic Bernoulli number identity first discovered by Euler:

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$$

It is natural to ask if other Bernoulli number identities can be obtained by substitution. For example, is there a function $f(x)$ which when substituted into (2) will yield the following quadratic identity?

$$\sum_{k=0}^{n+1} \binom{n+1}{k} B_k B_{n-k+1} = -(n+1)B_n - nB_{n+1} \tag{3}$$

The answer, not surprisingly, is yes. The surprise however is the choice for $f(x)$. It is clear that $f(x)$ should involve the Bernoulli numbers since (3) contains products of Bernoulli numbers. Therefore, a natural choice for $f(x)$ would be to set it equal to a Bernoulli polynomial, say $B_n(x)$. However, the reader will discover that substituting $f(x) = B_n(x)$ into (2) yields the trivial identity. The correct answer is $f(x) = (1-x)B_n(x)$.

The Bernoulli polynomials $B_n(x)$ give an example of an Appell sequence. As such, there are two equivalent definitions for $B_n(x)$: one via the exponential generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{4}$$

and the other as a polynomial sequence with the following properties: