Journal of Number Theory 137 (2014) 241–255

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Integral-valued polynomials over sets of algebraic integers of bounded degree

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ARTICLE INFO

Article history:
Received 7 January 2013
Received in revised form 3 November 2013
Accepted 4 November 2013
Available online 3 January 2014
Communicated by David Goss

MSC:
13B25
13F20
11C

Keywords:
Integer-valued polynomial
Algebraic integers with bounded degree
Prüfer domain
Polynomially dense subset
Integral closure
Pullback

ABSTRACT

Let K be a number field of degree n with ring of integers O_K. By means of a criterion of Gilmer for polynomially dense subsets of the ring of integers of a number field, we show that, if $h \in K[X]$ maps every element of O_K of degree n to an algebraic integer, then $h(X)$ is integral-valued over O_K, that is, $h(O_K) \subset O_K$. A similar property holds if we consider the set of all algebraic integers of degree n and a polynomial $f \in \mathbb{Q}[X]$: if $f(\alpha)$ is integral over $\mathbb{Z}$ for every algebraic integer α of degree n, then $f(\beta)$ is integral over $\mathbb{Z}$ for every algebraic integer β of degree smaller than n. This second result is established by proving that the integral closure of the ring of polynomials in $\mathbb{Q}[X]$ which are integer-valued over the set of matrices $M_n(\mathbb{Z})$ is equal to the ring of integral-valued polynomials over the set of algebraic integers of degree equal to n.

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http://dx.doi.org/10.1016/j.jnt.2013.11.007

1. Introduction

Let K be a number field of degree n over $\mathbb{Q}$ with ring of integers O_K. Given $f \in K[X]$ and $\alpha \in O_K$, the evaluation of $f(X)$ at α is an element of K. If $f(\alpha)$ is in O_K we say that $f(X)$ is integral-valued on α. If this condition holds for every $\alpha \in O_K$, we say that $f(X)$ is integral-valued over O_K. The set of such polynomials forms a ring, usually denoted by:

$$\mathrm{Int}(O_K) \doteqdot \{f \in K[X] \mid f(O_K) \subset O_K\}.$$

Obviously, $\mathrm{Int}(O_K) \supset O_K[X]$ and this is a strict containment (over $\mathbb{Z}$, consider $X(X-1)/2$). A classical problem regarding integral-valued polynomials is to find proper subsets S of O_K such that if $f(X)$ is any polynomial in $K[X]$ such that $f(s)$ is in O_K for all s in S then $f(X)$ is integral-valued over O_K. A subset S of O_K with this property is usually called a polynomially dense subset of O_K. For example, it is easy to see that cofinite subsets of O_K have this property. For a general reference of polynomially dense subsets and the so-called polynomial closure see [1] (see also the references contained in there). Obviously, for a polynomially dense subset S we have $\mathrm{Int}(S, O_K) \doteqdot \{f \in K[X] \mid f(S) \subset O_K\} = \mathrm{Int}(O_K)$ (in general we only have one containment). Gilmer gave a criterion which characterizes polynomially dense subsets of a Dedekind domain with finite residue fields [6]. His result was later elaborated by McQuillan in this way ([9]; we state it for the ring of integers of a number field): a subset S of O_K is polynomially dense in O_K if and only if, for every non-zero prime ideal P of O_K, S is dense in O_K with respect to the P-adic topology. By means of this criterion, we show here the following theorem.

Theorem 1.1. *Let K be a number field of degree n over $\mathbb{Q}$. Let $O_{K,n}$ be the set of algebraic integers of K of degree n. Then $O_{K,n}$ is polynomially dense in O_K.*

The previous problem concerns the integrality of the values of a polynomial with coefficients in a number field K over the set of algebraic integers of K. We also address here our interest to the study of the integrality of the values of a polynomial with rational coefficients over the set of algebraic integers of a proper finite extension of $\mathbb{Q}$, or, more in general, over a set of algebraic integers which lie in possibly infinitely many number fields, but of bounded degree. In this direction, Loper and Werner introduced in [8] the following ring of integral-valued polynomials:

$$\mathrm{Int}_{\mathbb{Q}}(O_K) \doteqdot \{f \in \mathbb{Q}[X] \mid f(O_K) \subset O_K\}.$$

This ring is the contraction to $\mathbb{Q}[X]$ of $\mathrm{Int}(O_K)$. It is easy to see that it is a subring of the usual ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subset \mathbb{Z}\}$. Moreover, this is always a strict containment: take any prime integer p such that there exists a prime