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Congruences for t-core partition functions

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Let $t \geqslant 2$ be an integer and $p \geqslant 5$ be a prime. We prove a conjecture on congruences for 2^t-core partition functions. We also find many new congruences for p-core partition functions when $5 \leqslant p \leqslant 47$.

1. Introduction

A partition of a positive integer n is any non-increasing sequence of positive integers whose sum is n. Given a partition $[\lambda] = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ of n, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$, the Ferrers–Young diagram of $[\lambda]$ is an array of nodes with λ_i nodes in the ith row. The (i, j) hook is the set of nodes directly below, together with the set of nodes directly to the right of the (i, j) nodes, as well as the (i, j) node itself. The hook number, denoted by $H(i, j)$, is the total number of nodes on the (i, j) hook.

For a positive integer t, a *t-core partition* of n is a partition of n in which none of the hook numbers are divisible by t. We denote by $a_t(n)$ the number of t-core partitions

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of n. We illustrate the Ferrers–Young diagram of the partition $4+3+1$ of 8 with hook numbers as follows:

$$\begin{array}{llll} \bullet^6 & \bullet^4 & \bullet^3 & \bullet^1 \\ \bullet^4 & \bullet^2 & \bullet^1 & \\ \bullet^1 & & & \end{array}$$

It is clear that if $t \geqslant 7$, then the partition $4+3+1$ of 8 is a t-core partition.

t-core partitions are of importance in combinatorial number theory. For example, Garvan, Kim and Stanton [5] used explicit statistics, called cranks, to prove the famous Ramanujan's congruences for the partition function $p(n)$: for all nonnegative integers n,

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Moreover, they proved that the generating function for $a_t(n)$ is

$$\sum_{n=0}^{\infty} a_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}. \tag{1}$$

One can refer to [11] for a different combinatorial proof. t-core partitions also play important roles in the study of modular representation of symmetric group S_n [10].

The purpose of this paper is to establish some congruence properties of $a_t(n)$. There are numerous beautiful results on congruences for $a_t(n)$. For instance, Garvan, Kim and Stanton [5] proved that if α is a positive integer and $\ell = 5, 7, 11$, then

$$a_\ell\big(\ell^\alpha n - \delta_\ell\big) \equiv 0 \pmod{\ell^\alpha}$$

for all nonnegative integers n, where $\delta_\ell = \frac{\ell^2-1}{24}$. Applying Ramanujan's congruences for the partition function $p(n)$ modulo powers of 5, 7, 11, Granville and Ono [6] showed that

$$a_{5^\alpha}\big(5^\alpha n - \delta_{5,\alpha}\big) \equiv 0 \pmod{5^\alpha},$$
$$a_{7^\alpha}\big(7^\alpha n - \delta_{7,\alpha}\big) \equiv 0 \pmod{7^{[\alpha/2]+1}},$$
$$a_{11^\alpha}\big(11^\alpha n - \delta_{11,\alpha}\big) \equiv 0 \pmod{11^\alpha},$$

where $\delta_{\ell,\alpha} = \frac{1}{24} \pmod{\ell^\alpha}$.

When t is a power of 2, arithmetic properties of $a_t(n)$ have been studied by the author in [3]. For 4-core, 8-core and 16-core partition functions, Hirschhorn, Kolitsch and