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Probabilistic properties of number fields

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ABSTRACT

In general a bound on number theoretic invariants under the Generalized Riemann Hypothesis (GRH) for the Dedekind zeta function of a number field K is much stronger than an unconditional one. In this article, we consider three invariants; the residue of $\zeta_K(s)$ at $s = 1$, the logarithmic derivative of Artin L-function attached to K at $s = 1$, and the smallest prime which does not split completely in K. We obtain bounds on them just as good as the bounds under GRH except for a density zero set of number fields.

1. Introduction

There are many results on number fields assuming Generalized Riemann Hypothesis (GRH). Unconditional results are a lot worse. In this paper, we show that even though one cannot improve the result on a single number field without GRH, if we consider a

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family of number fields, one can obtain bounds just as good as GRH bounds except for small exceptions.

Let K be a number field of degree n with d_K, the absolute value of the field discriminant of K. Let $\hat{K}$ be its Galois closure, and $Gal(\hat{K}/\mathbb{Q}) = G$. We call such a field K a G-field. Let $\zeta_K(s)$ be the Dedekind zeta function. We attach an Artin L-function $L(s,\rho) = \frac{\zeta_K(s)}{\zeta(s)}$, where ρ is an $(n-1)$-dimensional representation of G. We are interested in three invariants attached to K:

(1) $L(1,\rho)$, the residue of $\zeta_K(s)$ at $s = 1$;
(2) $\frac{L'}{L}(1,\rho) = \gamma_K - \gamma$, where γ_K is the Euler–Kronecker constant, given as follows: $\gamma_K = \frac{c_0}{c_{-1}}$, where $\zeta_K(s) = c_{-1}(s-1)^{-1} + c_0 + c_1(s-1) + \cdots$, and $\gamma = \gamma_{\mathbb{Q}}$;
(3) N_K, the smallest prime which does not split completely in a number field K.

In Section 2, we collect known bounds on $L(1,\rho)$, $\frac{L'}{L}(1,\rho)$ and N_K under GRH, and also unconditional bounds.

We fix a degree n and a transitive subgroup G of S_n and given $X > 0$, consider all number fields K, which are given by irreducible polynomials of degree n, $Gal(\hat{K}/\mathbb{Q}) = G$, and $d_K < X$, where d_K is the absolute value of the discriminant of K. It is an important problem to determine the number of such fields. Malle [19,20] gave a conjecture on the number of such fields. In Section 3, we review known results on Malle's conjecture.

If we assume the strong Artin conjecture for $L(s,\rho) = \zeta_K(s)/\zeta(s)$, we can apply the zero-density result of Kowalski and Michel [14] to show that except for small exceptions, $L(s,\rho)$ is zero-free for $[\alpha,1] \times [-(\log d_K)^2, (\log d_K)^2]$. Then we apply the result of Daileda [8] to obtain our result as in our earlier papers [5,4,6].

There are two difficulties to proceed: First, it may be possible that different $L(s,\rho)$ coincide, namely, $\zeta_{K_1}(s) = \zeta_{K_2}(s)$. In such a case we say that K_1 and K_2 are arithmetically equivalent. So we need to remove the repetition. Second, when ρ is reducible, in order to apply the zero-density result of Kowalski and Michel, we need a certain technical condition (4.1). Assuming Malle's conjecture, the strong Artin conjecture and condition (4.1), we obtain good bounds on $L(1,\rho)$, $\frac{L'}{L}(1,\rho)$ and N_K except for small exceptions:

Theorem 1.1. *Assume Malle's conjecture (Conjecture 3.1) and the strong Artin conjecture for $L(s,\rho)$. Assume condition (4.1) holds for G-fields. Let $L(X)$ be the set of all G-fields with $d_K \leqslant X$. Except for $O(X^{\frac{99}{100}a(G)+\epsilon})$ G-fields in $L(X)$, the following statements are true.*

(1) $\frac{1}{\log\log d_K} \ll_G L(1,\rho) \ll_G (\log\log d_K)^{n-1}$;
(2) $-\frac{16(n-1)}{1-\alpha}\log\log d_K + O_G(1) \leqslant \frac{L'}{L}(1,\rho) \leqslant \frac{16}{1-\alpha}\log\log d_K + O_G(1)$;
(3) $N_K \ll_G (\log d_K)^{\frac{16}{(1-\alpha)A}}$,

where α and $a(G) > 0$ depend only on G and $A = \sup_{\lambda \geqslant 0} \frac{1-\frac{1}{n-1}e^{-\lambda}}{\lambda}$.