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E-sequences and the Stone–Weierstrass Theorem

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ABSTRACT

For discrete valuation domain V with finite residue field, we define a variation of the V.W.D.W.O. sequence of Cahen and Chabert in order to construct V-bases for the algebra of even integer-valued polynomials on V and the module of odd integer-valued polynomials on V. Using these bases, we prove a version of the Stone–Weierstrass Theorem for V, namely, that every even (respectively odd) continuous function on the completion $\hat{V}$ can be approximated by means of even (respectively odd) integer-valued polynomials on V. Using these approximations, we give series expansions for all even (respectively odd) continuous functions on $\hat{V}$, analogous to results of Mahler for the p-adic integers.

1. Introduction

For an integral domain D with field of fractions K, the set of integer-valued polynomials on D is defined to be $\operatorname{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}$; it is in fact a D-algebra containing $D[X]$. Interest in $\operatorname{Int}(\mathbb{Z})$ is classical, with the definition of the polynomials $\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$ dating back to Newton and Gregory in the seventeenth century [2]. In 1919 Polyá [9] and Ostrowski [8] studied free bases for the D-module $\operatorname{Int}(D)$ for the case where D is the ring of algebraic integers in a number field K, and in 1936 Skolem [12] studied the ring structure of $\operatorname{Int}(\mathbb{Z})$. Renewed interest in rings of integer-valued polynomials began in the 1980's after Brizolis [1] showed that the ring $\operatorname{Int}(\mathbb{Z})$ is a Prüfer domain of Krull dimension two. For an extensive bibliography on integer-valued polynomials through 1996, see [2].

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The Stone–Weierstrass Theorem for discrete valuation domains with finite residue field says that continuous integer-valued *functions* can be approximated by integer-valued *polynomials*. Dieudonné [3] proved a version of this theorem for the p-adic integers, later generalized by Kaplansky [5] and Mahler [6]. Mahler gave an explicit expansion of functions continuous on $\hat{\mathbb{Z}}_p$ using series of the basis polynomials $\binom{X}{n}$ (e.g., Example 1 below). These results were generalized to arbitrary discrete valuation domains with finite residue fields by Cahen and Chabert [2], using what they call V.W.D.W.O. sequences (Definition 1 below).

Among the many interesting exercises in Polyá and Szegö's classic *Problems and Theorems in Analysis, II* [10], problems 88 and 89 of the number theory section claim that the sequences of polynomials $E_n = \frac{X}{n}\binom{X+n-1}{2n-1}$ (with $E_0 = 1$) and $O_n = \binom{X+n-1}{2n-1}$ form $\mathbb{Z}$-bases for the even polynomials and the odd polynomials, respectively, in $\mathrm{Int}(\mathbb{Z})$. Indeed, these same sequences form $\hat{\mathbb{Z}}_p$-bases for the even polynomials and the odd polynomials, respectively, in $\mathrm{Int}(\hat{\mathbb{Z}}_p)$ for any rational prime p. This leads naturally to the question of whether one can adapt V.W.D.W.O. sequences to generalize these polynomial sequences $\{E_n\}_{n\geqslant 0}$ and $\{O_n\}_{n\geqslant 1}$ to work for arbitrary discrete valuation domains with finite residue fields. This we do in Section 2, introducing the notion of "extendable" sequences (E-sequences) in Definition 2. In Theorem 1 we show that most discrete valuation domains with finite residue fields have an E-sequence (with restrictions only in case the residue field has even order). For discrete valuation domain V with an E-sequence, we construct a polynomial sequence $\{E_n\}_{n\geqslant 0}$ forming a V-basis for the algebra $\mathrm{Int}^e(V)$ of even integer-valued polynomials on V (Theorem 2) and a polynomial sequence $\{O_n\}_{n\geqslant 1}$ forming a V-basis for the module $\mathrm{Int}^o(V)$ of odd integer-valued polynomials on V (Theorem 3). As an application, we show that the ring $\mathrm{Int}^e(V)$ is not factorial and compute a lower bound on its elasticity (Corollary 2).

In Section 3 we use the bases $\{E_n\}_{n\geqslant 0}$ and $\{O_n\}_{n\geqslant 1}$ to prove the Stone–Weierstrass Theorem for even continuous functions (Theorem 5) and odd continuous functions (Theorem 7) for discrete valuation rings with E-sequences. Using these approximation theorems, we derive series expansions for even continuous functions (Theorem 6) and odd continuous functions (Theorem 8) on the completion of a discrete valuation domain with E-sequence in terms of the polynomial bases $\{E_n\}_{n\geqslant 0}$ and $\{O_n\}_{n\geqslant 1}$. Finally, decomposing a function into even and odd parts, we combine these series to get an analog of a Fourier series expansion for continuous functions on the completion (Corollary 3).

Properties of even and odd functions are used in many areas of mathematics, for example the proofs of the results on the digamma function in [7] and those for results on the L-polynomials in [11] used properties of even and odd functions, respectively. Even and odd functions are an interesting class of functions in their own right; see [4] and [13]. Nonetheless, little is known about even and odd functions defined in abstract settings, such as over discrete valuation domains. In this paper, we show the possibility of expanding even and odd continuous function on the completion $\hat{V}$ as a series of even and odd, respectively, integer-valued polynomials. Note that this intuitive result does not automatically follow from the corresponding result for continuous functions and integer-valued polynomials in general. In Example 1, we give a simple example of an even continuous function on the 5-adic integers $\mathbb{Z}_5$ whose Mahler expansion using the sequence $\binom{X}{n}$ of polynomials from [2, Theorem II.2.7] is not an expansion by even polynomials.

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2. E-sequences

Throughout this paper, we let V be a discrete valuation domain with field of fractions K, valuation v, maximal ideal $\mathfrak{m}$, and finite residue field $V/\mathfrak{m}$ of order q. We begin by recalling the notion of a *very well distributed and well ordered sequence*, an analog to the set of natural numbers in the ring of p-adic integers [2, Definition II.2.1].

Definition 1. A sequence $\{u_n\}_{n\geqslant 0}$ of elements of V is said to be a V.W.D.W.O. sequence if, for all non-negative integers n and m, $v(u_n - u_m) = v_q(n-m)$, the largest power of q that divides $n-m$.