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Congruences on the Bell polynomials and the derangement polynomials

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ABSTRACT

In this note, by the umbral calculus method, the Sun and Zagier congruences involving the Bell numbers and the derangement numbers are generalized to the polynomial cases. Some special congruences are also presented.

1. Introduction

It is well known that the first and second kind Stirling numbers $s(m, j)$ and $S(m, j)$ [11] are defined respectively by

$$x(x-1)\cdots(x-m+1)=\sum_{j=0}^{m}s(m,j)x^j, \tag{1.1}$$

$$\sum_{j=0}^{m}S(m,j)x(x-1)\cdots(x-j+1)=x^m. \tag{1.2}$$

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The Bell polynomials $\{\mathcal{B}_m(x)\}_{m\geqslant 0}$ are defined by

$$\mathcal{B}_m(x)=\sum_{j=0}^{m}S(m,j)x^j.$$

It is clear that $\mathcal{B}_m(1)$ is the m-th Bell number, denoted by B_m, counting the number of partitions of $[m]=\{1,2,\ldots,m\}$ (with $B_0=1$). The Bell polynomials $\mathcal{B}_m(x)$ satisfy the recurrence

$$\mathcal{B}_{m+1}(x)=x\sum_{j=0}^{m}\binom{m}{j}\mathcal{B}_j(x). \tag{1.3}$$

The derangement polynomials $\{\mathcal{D}_m(x)\}_{m\geqslant 0}$ are defined by

$$\mathcal{D}_m(x)=\sum_{j=0}^{m}\binom{m}{j}j!(x-1)^{m-j}.$$

Clearly, $\mathcal{D}_m(1)=m!$ and $\mathcal{D}_m(0)$ is the m-th derangement number, denoted by D_m, counting the number of fixed-point-free permutations on $[m]$ (with $D_0=1$). The derangement polynomials $\mathcal{D}_m(x)$, also called x-factorials of m, have been considerably investigated by Eriksen, Freij and Wästlund [4], Sun and Zhuang [16]. They obey the recursive relation

$$\mathcal{D}_m(x)=m\mathcal{D}_{m-1}(x)+(x-1)^m. \tag{1.4}$$

Recently, Sun [12] discovered experimentally that for a fixed positive integer m the sum $\sum_{k=0}^{p-1}B_k/(-m)^k$ modulo a prime p not dividing m is independent of p, a typical case being

$$\sum_{k=0}^{p-1}\frac{B_k}{(-8)^k}\equiv -1853 \pmod p \quad \text{for all primes } p\neq 2.$$

Later, Sun and Zagier [15] confirmed this conjecture and proved the nice result.

Theorem 1.1. *For any integer $m\geqslant 1$ and any prime $p\nmid m$, there holds*

$$(-x)^m\sum_{k=1}^{p-1}\frac{\mathcal{B}_k(x)}{(-m)^k}\equiv(-x)^p\sum_{k=0}^{m-1}\frac{(m-1)!}{k!}(-x)^k \pmod{p\mathbb{Z}_p[x]},$$

where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Particularly, the case $x=1$ generates

$$\sum_{k=1}^{p-1}\frac{B_k}{(-m)^k}\equiv(-1)^{m-1}D_{m-1} \pmod p. \tag{1.5}$$

Here for two polynomials $P(x),Q(x)\in\mathbb{Z}_p[x]$, by $P(x)\equiv Q(x) \pmod{p\mathbb{Z}_p[x]}$ we mean that the corresponding coefficients of $P(x)$ and $Q(x)$ are congruent modulo p.

In this note, we establish a more general result of Sun and Zagier's congruence.