Journal of Number Theory 132 (2012) 3056–3069

Contents lists available at SciVerse ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the GCD-s of k consecutive terms of Lucas sequences ☆

L. Hajdu *, M. Szikszai

University of Debrecen, Institute of Mathematics, H-4010 Debrecen, P.O. Box 12, Hungary

ARTICLE INFO

Article history:
Received 17 January 2012
Revised 4 March 2012
Accepted 1 May 2012
Available online 13 August 2012
Communicated by David Goss

Keywords:
Lucas sequences
Greatest common divisor
Divisibility sequences
Pillai's problem

ABSTRACT

Let $u = (u_n)_{n=0}^{\infty}$ be a Lucas sequence, that is a binary linear recurrence sequence of integers with initial terms $u_0 = 0$ and $u_1 = 1$. We show that if k is large enough then one can find k consecutive terms of u such that none of them is relatively prime to all the others. We even give the exact values g_u and G_u for each u such that the above property first holds with $k = g_u$; and that it holds for all $k \geqslant G_u$, respectively. We prove similar results for Lehmer sequences as well, and also a generalization for linear recurrence divisibility sequences of arbitrarily large order. On our way to prove our main results, we provide a positive answer to a question of Beukers from 1980, concerning the sums of the multiplicities of 1 and -1 values in non-degenerate Lucas sequences. Our results yield an extension of a problem of Pillai from integers to recurrence sequences, as well.

1. Introduction

Let $u = (u_n)_{n=0}^{\infty}$ be a Lucas sequence, that is a binary linear recurrence sequence of integers with initial terms $u_0 = 0$ and $u_1 = 1$. The investigation of the divisibility properties of the terms of such sequences, or more generally, linear recurrence sequences, has a very long history, and a huge literature. Here we only mention a few of the several important and interesting directions, considered by many authors.

One of the most important questions concerns the existence of primitive prime divisors of the terms of Lucas sequences. After several results yielding partial answers to this problem, Bilu, Hanrot

☆ Research supported in part by the OTKA grants K75566, K100339, NK101680, and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project. The project is implemented through the New Hungary Development Plan, cofinanced by the European Social Fund and the European Regional Development Fund.

* Corresponding author.
E-mail addresses: hajdul@math.unideb.hu (L. Hajdu), szikszaimarton@gmail.com (M. Szikszai).

0022-314X/$ – see front matter
http://dx.doi.org/10.1016/j.jnt.2012.05.022

and Voutier [9] could completely settle the question. Since there are so many related results, instead of trying to summarize them, for the history of the problem we just refer the reader to [9] and the references given there.

Another important problem which has been closely investigated is to characterize the so-called divisibility sequences. That is, describe all linear recurrence sequences u_n (now of arbitrary order) such that $u_i \mid u_j$ whenever $i \mid j$. After certain partial results of Hall [24] and Ward [45], the complete description of such sequences has been provided by Bézivin, Pethő and van der Poorten [8]; see also the paper of Győry and Pethő [21]. There are also important related results of Horák and Skula [25] and Schinzel [41], concerning so-called strong divisibility sequences.

The next topic we mention concerns the investigation of the property $n \mid u_n$ for $n \geqslant 1$; i.e. the determination of terms being divisible by their indices. For related results see e.g. the papers of Smyth [44], Győry and Smyth [22] and Alba González et al. [1], and the references therein.

Another problem is to find the prime terms of the sequences studied, or at least prove that there are only finitely many such terms. For related results and references we refer to the book of Guy [20], p. 17, and the papers of Graham [19], Knuth [27], Wilf [47] and Dubickas et al. [17], and the references given there.

There are several results concerning the problem when a term or the product of terms of a sequence u (or even of more sequences) is a perfect power; see e.g. the book of Shorey and Tijdeman [43] and the papers of Bremner and Tzanakis [11–13], Luca and Walsh [35], Kiss [26], Brindza, Liptai and Szalay [14], and Luca and Shorey [32–34], and the references there.

Finally, we mention a problem which is not closely related, but on the one hand deeply investigated, and on the other hand, also important from the viewpoint of the present paper. This problem is the question of zero-multiplicity (or more generally the ω-multiplicity) of linear recurrence sequences. That is, given such a sequence $u = (u_n)_{n=0}^{\infty}$, we are interested in the number of solutions of $u_n = 0$ (or more generally of $u_n = \omega$ with some given number ω). Further, in case of infinitely many solutions, we would like to know the structure of solutions. After the fundamental results of Skolem, Mahler and Lech [30] several much more general results have appeared; see e.g. the papers of Ward [46], Kubota [28,29], Beukers [5,6], Beukers and Tijdeman [7], Brindza, Pintér and Schmidt [15], Allen [2,3], Amoroso and Viada [4] and the references given there.

In this paper we consider a property of linear recurrence sequences which is strongly related to the above ones. To set the problem, first we recall a question considered by Pillai [39]: is it true that for any $k \geqslant 2$ one can find k consecutive integers such that none of them is relatively prime to all the others? Pillai [39] himself proved that this is not true for $2 \leqslant k \leqslant 16$, but holds for $17 \leqslant k \leqslant 430$. The question has been completely answered to the affirmative by Brauer [10]. Later, the original problem has been extended and generalized into several directions. For related results, see e.g. the papers of Caro [16], Saradha and Thangadurai [40] and Hajdu and Saradha [23] and the references there. In particular, Ohtomo and Tamari [38] have extended the original problem to arithmetic progressions, i.e. one considers k consecutive terms of an arithmetic progression, rather than k consecutive integers.

In this paper we extend Pillai's problem to linear recurrence sequences. More precisely, we consider the following problem, and also some of its generalizations. Let $u = (u_n)_{n=0}^{\infty}$ be a non-degenerate Lucas sequence. (For exact definitions and notation see Section 2.) Let g_u be the smallest integer such that for $k = g_u$, one can find k consecutive terms in u such that none of these terms is relatively prime to all the others. Similarly, let G_u be the smallest integer k_0 such that for any $k \geqslant k_0$ one can find k consecutive terms in u such that none of these terms is relatively prime to all the others. Note that *a priori* it is not known that g_u and G_u exist. However, if they both exist, then we obviously have $g_u \leqslant G_u$. We prove that for any non-degenerate Lucas sequence u, both g_u and G_u exist, and further, we calculate the exact values of these numbers for each u (see Theorem 1). On our way to prove this result, we provide a positive answer to a question of Beukers [5], concerning the sums of the multiplicities of the values 1 and -1 in non-degenerate Lucas sequences (see Corollary 10). Just for curiosity, we also mention that as a special case we obtain that among any 24 consecutive Fibonacci numbers one of them is always coprime to all the others, however, it is possible to find 25 consecutive Fibonacci numbers lacking this property. The index n_0 of the first term of 25 such numbers where this phenomenon first occurs is $n_0 = 208\,569\,474$.

We provide a similar result also for Lehmer sequences (cf. Theorem 3).