Journal of Number Theory 130 (2010) 2480–2527

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the lifting of elliptic cusp forms to cusp forms on quaternionic unitary groups

Shunsuke Yamana [1]

Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

ARTICLE INFO

Article history:
Received 15 February 2008
Revised 25 July 2009
Available online 8 July 2010
Communicated by Stephen Rallis

MSC:
11F55
11F30

Keywords:
Quaternion algebras
Ikeda lifting
Maass lifting

ABSTRACT

Let H be a definite quaternion algebra over $\mathbb{Q}$ with discriminant D_H and R a maximal order of H. We denote by G_n a quaternionic unitary group and put $\Gamma_n = G_n(\mathbb{Q}) \cap \mathrm{GL}_{2n}(R)$. Let $S_\kappa(\Gamma_n)$ be the space of cusp forms of weight κ with respect to Γ_n on the quaternion half-space of degree n. We construct a lifting from primitive forms in $S_k(\mathrm{SL}_2(\mathbb{Z}))$ to $S_{k+2n-2}(\Gamma_n)$ and a lifting from primitive forms in $S_k(\Gamma_0(d))$ to $S_{k+2}(\Gamma_2)$, where d is a factor of D_H. These liftings are generalizations of the Maass lifting investigated by Krieg.

To my father

0. Introduction

The purpose of this paper is to construct a lifting that associates to an elliptic cusp form a cusp form on a quaternionic unitary group. This is a quaternionic modular analogue of the liftings constructed by Ikeda [14,15]. In a similar fashion, Ikeda constructed, from an elliptic cusp form, a Siegel cusp form in [14] and a hermitian cusp form in [15].

Let us describe our results. Let H be a definite quaternion algebra over $\mathbb{Q}$ and $^\iota$ the main involution of H. Fix a maximal order R of H. Let $\mathbb{H} = H \otimes_\mathbb{Q} \mathbb{R}$, $H_p = H \otimes_\mathbb{Q} \mathbb{Q}_p$ and $R_p = R \otimes_\mathbb{Z} \mathbb{Z}_p$. Put $x^* = {}^t x^\iota$ for $x \in \mathrm{M}_n(H)$.

E-mail address: yamana@sci.osaka-cu.ac.jp.

[1] We would like to thank Prof. Ikeda for suggesting this problem and for his valuable advice and guidance. Parts of this work were done while the author was visiting Aachen University. The author thanks the staffs of Aachen University, especially Prof. Krieg, for their hospitality and encouragement. The author is supported by the Grant-in-Aid for JSPS Fellows.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2010.03.020

Let G_n be a connected algebraic group defined over $\mathbb{Q}$ whose group of $\mathbb{Q}$-valued points is given by

$$G_n(\mathbb{Q}) = \left\{\alpha \in \mathrm{SL}_{2n}(H) \;\middle|\; \alpha \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix} \alpha^* = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}\right\}.$$

The modular group is defined to be $\Gamma_n = \mathrm{GL}_{2n}(R) \cap G_n(\mathbb{Q})$.

For a ring $\mathcal{O}$ with involution ${}^\iota$, we put $S_n(\mathcal{O}) = \{x \in \mathrm{M}_n(\mathcal{O}) \mid {}^t x^\iota = x\}$. The quaternion upper half-space of degree n is defined by

$$\mathfrak{H}_n = \left\{Z = X + \sqrt{-1}Y \in S_n(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \;\middle|\; X \in S_n(\mathbb{H}),\ 0 < Y \in S_n(\mathbb{H})\right\}.$$

For any $\mathbb{Q}$-algebra D, let ν, $\tau : \mathrm{M}_n(H \otimes_{\mathbb{Q}} D) \to D$ be the reduced norm and the reduced trace on $\mathrm{M}_n(H \otimes_{\mathbb{Q}} D)$ respectively. Put $\lambda = \frac{1}{2}\tau$. We define a polynomial map $\mathrm{Paf} : S_n(H) \to \mathbb{Q}$, using the relations

$$\mathrm{Paf}(\mathbf{1}_n) = 1, \qquad \mathrm{Paf}(X)^2 = \nu(X), \quad X \in S_n(H).$$

Let κ be an even integer. For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_n(\mathbb{R})$, $Z \in \mathfrak{H}_n$ and a function F on $\mathfrak{H}_n$, we put

$$\alpha Z = (aZ + b)(cZ + d)^{-1}, \qquad F|_\kappa \alpha(Z) = \nu(cZ + d)^{-\kappa/2} F(\alpha Z).$$

When $n \geqslant 2$, a modular form F of weight κ is a holomorphic function on $\mathfrak{H}_n$ which satisfies $F|_\kappa \gamma = F$ for every $\gamma \in \Gamma_n$. Put

$$T_n = \left\{h \in S_n(H) \;\middle|\; \lambda(h\beta) \in \mathbb{Z} \text{ for every } \beta \in S_n(R)\right\}$$

and let T_n^+ denote the set of positive definite elements of T_n. A modular form F is called a cusp form if it has a Fourier expansion of the form

$$F(Z) = \sum_{h \in T_n^+} A_F(h) \mathbf{e}\big(\lambda(hZ)\big)$$

(cf. Remark 1.3). Let $S_\kappa(\Gamma_n)$ be the space of cusp forms on $\mathfrak{H}_n$ of weight κ.

Krieg systematically developed the theory of modular forms on $\mathfrak{H}_n$ in [22], but he makes the following assumptions:

(I) H is the Hurwitz quaternion, i.e., H has a basis $\{1, i, j, k\}$ over $\mathbb{Q}$ such that $k = ij = -ji$, $i^2 = j^2 = -1$;
(II) R is the Hurwitz order, i.e., $R = \mathbb{Z}[i, j, k, \frac{1+i+j+k}{2}]$.

The present paper investigates modular forms with respect to the group Γ_n which comes from an arbitrary definite quaternion algebra over $\mathbb{Q}$.

Fix a rational prime p. The Siegel series attached to $h \in T_n$ is defined by

$$b_p(h, s) = \sum_{\beta \in S_n(H_p)/S_n(R_p)} \mathbf{e}_p\big(-\lambda(h\beta)\big) \nu[\beta]^{-s/2},$$

where $\nu[\beta] = [\beta R_p^n + R_p^n : R_p^n]^{1/2}$.

Let D_H be the discriminant of H. Put $D_h = D_H^{[n/2]} \mathrm{Paf}\, h$. If h is nondegenerate, then there exists a polynomial $F_{p,h}$ such that