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Automorphism group of parafermion vertex operator algebras

Qing Wang [1]

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

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ABSTRACT

The automorphism group of parafermion vertex operator algebra associated with the irreducible highest weight module for the affine Kac–Moody algebra $A_1^{(1)}$ was easily determined since the Virasoro primary vector of weight 3 in this parafermion vertex operator algebra is unique up to a scalar. However, it is highly nontrivial to determine the automorphism group of parafermion vertex operator algebra associated with the irreducible highest weight module for the affine Kac–Moody algebra $A_n^{(1)}$ with the rank $n \geq 2$. As the first step, in this paper, we determine the full automorphism group of parafermion vertex operator algebra associated with the irreducible highest weight module for the affine Kac–Moody algebra $A_2^{(1)}$, which shows the idea for a complete determination for the full automorphism group of the parafermion vertex operator algebra associated with the irreducible highest weight module for any affine Kac–Moody algebra.

1. Introduction

It is well known that vertex operator algebras originated from the famous "monstrous moonshine" conjectures (see [23,4]). Central to these conjectures is the moonshine module $V^\natural$, constructed in [14], it is an important example of a vertex operator algebra. Famous monster simple group is the full automorphism group of the moonshine vertex operator algebra $V^\natural$ [3,14]. In view of this, to study the automorphism group of a vertex operator algebra is not only interesting in the group theory, but also useful in the study of the vertex operator algebra itself. However, there is no uniform method to determine the automorphism group of the vertex operator algebra. Some depend on the structures of Lie algebras or lattices or codes which were used to construct the vertex operator algebras.

Parafermion vertex operator algebras originated from Z-algebras [19–21] which were introduced in the course of constructing the integrable highest weight modules for affine Kac–Moody algebras. Recently, a series of papers are devoted to the study of the parafermion vertex operator algebras (see [6–8,10,11,2,9,12,1]). Let $k \geq 2$ be a positive integer, and let $\mathcal{L}(k, 0)$ be the level k integrable highest weight module with the

E-mail address: qingwang@xmu.edu.cn.

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highest weight zero for affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ associated with a finite dimensional simple Lie algebra $\mathfrak{g}$. Then $\mathcal{L}(k,0)$ contains the Heisenberg vertex operator subalgebra $M(k)$ generated by the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The commutant $K(\mathfrak{g},k)$ of $M(k)$ in $L(k,0)$ is the parafermion vertex operator algebra. In [10], we have determined the generators of the parafermion vertex operator algebra $K(\mathfrak{g},k)$ associated with any finite dimensional simple Lie algebra $\mathfrak{g}$. Among the obtained results in the structure theory and representation theory of the parafermion vertex operator algebras, we see the important role of $K(sl_2,k)$ played in the general parafermion vertex operator algebra $K(\mathfrak{g},k)$. Thus, we always start from the study of the parafermion vertex operator algebra $K(sl_2,k)$.

As we mentioned, the automorphism group of $K(sl_2,k)$ was easily determined since the Virasoro primary vector of weight 3 in $K(sl_2,k)$ is unique up to a scalar. However, it is not the case for the parafermion vertex operator algebra $K(sl_n,k)$ with $n \geq 3$ and other parafermion vertex operator algebras $K(\mathfrak{g},k)$ with $\mathfrak{g}$ being of type B, C, D, E, F, G. This is the first paper in a series to study the automorphism group of the parafermion vertex operator algebras. We start from the nontrivial case for determination of the automorphism group of the parafermion vertex operator algebra $K(sl_3,k)$. Since the weight one subspace of a parafermion vertex operator algebra is always zero, so we cannot use the Lie algebra structure on the weight one space to study the automorphism group of a parafermion vertex operator algebra. It is natural to turn the attention to weights two and three subspaces. In fact, the parafermion vertex operator algebras are generated by certain weights two and three vectors. As in the moonshine vertex operator algebra, the weight two subspace of a parafermion vertex operator algebra forms a commutative nonassociative algebra. Since there is no general structure and representation theory for this kind of algebra, one does not know how to deal with the weight two subspace effectively. For the weight three subspace, it is not even known what kind of algebraic structure induced from the structure of vertex operator algebra. Thus, we deal with the problem of determining the automorphism group of $K(sl_3,k)$ from the information of the generators and the relations among them. We first determine the action of the automorphism map on the generators ω_α in the weight two subspace for the positive roots α of Lie algebra sl_3. We prove that for an automorphism map σ of $K(sl_3,k)$, there exists a root system automorphism τ of sl_3, such that $\sigma(\omega_\alpha) = \omega_{\tau(\alpha)}$ for the positive roots α of Lie algebra sl_3. Then we use this fact to determine the action of the automorphism map on the generators W_α^3 in the weight three subspace for the positive roots α of Lie algebra sl_3. We obtain this by heavily using the integrable representation theory of the affine Kac–Moody algebras. We want to point out that it seems that the methods we used to determine the action of the automorphism map on the generators ω_α in the weight two subspace can be applied to the determination for the automorphism group of general parafermion vertex operator algebras $K(\mathfrak{g},k)$. However, it may encounter a calculation of a system of nonlinear equations with t indeterminate and a parameter k, where t is the number of positive roots of simple Lie algebra $\mathfrak{g}$. Unfortunately, we cannot solve it. However, we believe that the methods we used here shine some light on investigation of the automorphism group of general parafermion vertex operator algebras.

The paper is organized as follows. In Section 2, we fix the setting and recall the definition and some results on the parafermion vertex operator algebra $K(sl_3,k)$. In Section 3, we determine the full automorphism group of the parafermion vertex operator algebra $K(sl_3,k)$.

We expect the reader to be familiar with the elementary theory of vertex operator algebras as found, for example, in [14] and [18].

2. Parafermion vertex operator algebras $K(sl_3,k)$

In this section, we recall from [10] some basic results on the parafermion vertex operator algebra associated with the irreducible highest weight module for the affine Kac–Moody algebra $A_2^{(1)}$ of level k with $k \geq 2$ being an integer. First we recall the notion of the parafermion vertex operator algebra.