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Capturing Goodwillie's derivative

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ABSTRACT

Recent work of Biedermann and Röndigs has translated Goodwillie's calculus of functors into the language of model categories. Their work focuses on symmetric multilinear functors and the derivative appears only briefly. In this paper we focus on understanding the derivative as a right Quillen functor to a new model category. This is directly analogous to the behaviour of Weiss's derivative in orthogonal calculus. The immediate advantage of this new category is that we obtain a streamlined and more informative proof that the n-homogeneous functors are classified by spectra with a Σ_n-action. In a later paper we will use this new model category to give a formal comparison between the orthogonal calculus and Goodwillie's calculus of functors.

1. Introduction

Goodwillie's calculus of homotopy functors is a highly successful method of studying equivalence-preserving functors, often with source and target either spaces or spectra. The original development is given in the three papers by Goodwillie [9–11], motivated by the study of Waldhausen's algebraic K-theory of a space. A family of related theories grew out of this work; our focus is on the homotopy functor calculus and (to a lesser extent in this paper) the orthogonal calculus of Weiss [23]. The orthogonal calculus was developed to study functors from real inner-product spaces to topological spaces, such as $BO(V)$ and $TOP(V)$.

The model categorical foundations for the homotopy functor calculus and the orthogonal calculus have been established; see Biedermann–Chorny–Röndigs [5], Biedermann–Röndigs [6] and Barnes–Oman [3]. However, we have found these to be incompatible. Most notably, the symmetric multilinear functors of Goodwillie appear to have no analogue in the theory of Weiss. In this paper, we re-work the classification results of Goodwillie to make it resemble that of the orthogonal calculus. In a subsequent paper [2], we will

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use this similarity to give a formal comparison between the orthogonal calculus and Goodwillie's calculus of functors.

This re-working marks a substantial difference from the existing literature on model structures (or infinity categories) for Goodwillie calculus, as they follow the pattern of Goodwillie's work in a variety of different contexts (see also Pereira [21] or Lurie [16]). Our setup takes a more equivariant perspective and has the advantage of using one less adjunction and fewer categories than those of [6] and [11]. In detail, we construct a new category ($\Sigma_n \ltimes (\mathcal{W}_n \mathrm{Top})$, Section 4.2) which will be the target of an altered notion of the derivative over a point (diff_n, Section 6.1). This approach simplifies the classification of homogeneous functors in terms of spectra with Σ_n-action, whilst retaining Goodwillie's original classification at the level of homotopy categories, see Theorem 6.7. It also provides a new characterisation of the n-homogeneous equivalences, see Lemma 6.5 and clarifies some important calculations, see Examples 6.8 and 6.9.

1.1. *Recent history and context*

What the family of functor calculi have most in common is that they associate, to an equivalence-preserving functor F, a tower (the Taylor tower of F) of functors

$$\begin{array}{ccccccccc} & & D_nF & & D_{n-1}F & & & & D_1F & & \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{P}_nF & \longrightarrow & \mathrm{P}_{n-1}F & \longrightarrow & \cdots & \longrightarrow & \mathrm{P}_1F & \longrightarrow & P_0F \end{array}$$

where the P_nF have a kind of n-polynomial property, and for nice functors, the inverse limit of the tower, denoted $\mathrm{P}_\infty F$, is equivalent to F. The layers of the tower, D_nF, are then analogous to purely-n-polynomial functors – called n-homogeneous. Fig. 1 represents Goodwillie's classification of (finitary) n-homogeneous functors in terms of spectra with Σ_n-action. This classification is phrased in terms of three equivalences of homotopy categories.

$$\mathrm{Ho}(n\text{-homog-Fun}(C,\mathrm{Top})) \underset{\longleftarrow}{\overset{[11,\ \S 2]}{\longrightarrow}} \mathrm{Ho}(n\text{-homog-Fun}(C,\mathrm{Sp})) \underset{\longleftarrow}{\overset{[11,\ \text{Thm. } 3.5]}{\longrightarrow}} \mathrm{Ho}(\text{Symm-Fun}(C^n,\mathrm{Sp})_{ml}) \underset{\longleftarrow}{\overset{[11,\ \S 5]}{\longrightarrow}} \mathrm{Ho}(\Sigma_n \circlearrowleft \mathrm{Sp})$$

Fig. 1. Goodwillie's classification.

Here, "Ho" indicates that we are working with homotopy categories, n-homog-Fun(A, B) is the category of n-homogeneous functors from A to B, C is either spectra (Sp) or spaces (Top) and $\Sigma_n \circlearrowleft \mathrm{Sp}$ denotes (Bousfield–Friedlander) spectra with an action of Σ_n. The category $(\text{Symm-Fun}(C^n, \mathrm{Sp})_{ml})$ consists of symmetric multi-linear functors of n-inputs: those F with $F(X_1, \ldots, X_n) \cong F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for $\sigma \in \Sigma_n$ and which are degree 1-polynomial in each input.

Goodwillie, in [11], suggested that his classification would be well-served by being revised using the structure and language of model categories and hence phrased in terms of Quillen equivalences. For the homotopy functor calculus, Biedermann, Chorny and Röndigs [5] and Biedermann and Röndigs [6] completed Goodwillie's recommendation. For simplicial functors with fairly general target and domain, they follow the same pattern as Goodwillie's paper [11]. This classification involves several intermediate categories, similar to Fig. 1. In Fig. 2, $\mathcal{S}$ denote based simplicial sets, $\mathcal{S}^f$ denotes finite based simplicial sets and