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Polynomial growth and star-varieties

Daniela La Mattina [a,1], Fabrizio Martino [b,*,2]

[a] *Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123 Palermo, Italy*
[b] *IMECC, UNICAMP, Sérgio Buraque de Holanda 651, 13083-859 Campinas, SP, Brazil*

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ABSTRACT

Let $\mathcal{V}$ be a variety of associative algebras with involution over a field F of characteristic zero and let $c_n^*(\mathcal{V})$, $n = 1, 2, \ldots$, be its $*$-codimension sequence. Such a sequence is polynomially bounded if and only if $\mathcal{V}$ does not contain the commutative algebra $F \oplus F$, endowed with the exchange involution, and M, a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices. Such algebras generate the only varieties of $*$-algebras of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety is polynomially bounded. In this paper we completely classify all subvarieties of the $*$-varieties of almost polynomial growth by giving a complete list of finite dimensional $*$-algebras generating them.

1. Introduction

Let A be an associative algebra with involution ($*$-algebra) over a field F of characteristic zero and let $c_n^*(A)$, $n = 1, 2, \ldots$, be its sequence of $*$-codimensions.

Recall that $c_n^*(A)$, $n = 1, 2, \ldots$, is the dimension of the space of multilinear polynomials in n $*$-variables in the corresponding relatively free algebra with involution of countable rank. In case A satisfies a nontrivial identity, it was proved in [9] that, as in the ordinary case, $c_n^*(A)$ is exponentially bounded.

Given a variety of $*$-algebras $\mathcal{V}$, the growth of $\mathcal{V}$ is the growth of the sequence of $*$-codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}^*(A)$.

In this paper we are interested in varieties of polynomial growth, i.e., varieties of $*$-algebras such that $c_n^*(\mathcal{V}) = c_n^*(A)$ is polynomially bounded.

In such a case, if A is an algebra with 1, in [21] it was proved that

$$c_n^*(A) = qn^k + O(n^{k-1})$$

* Corresponding author.
E-mail addresses: daniela.lamattina@unipa.it (D. La Mattina), fabmartino@ime.unicamp.br (F. Martino).

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is a polynomial with rational coefficients. Moreover its leading term satisfies the inequalities

$$\frac{1}{k!} \leq q \leq \sum_{i=0}^{k} 2^{k-i} \frac{(-1)^i}{i!}.$$

In case of polynomial growth, the following characterization was given in [8]: a variety $\mathcal{V}$ has polynomial growth if and only if $\mathcal{V}$ does not contain the commutative algebra $F \oplus F$, endowed with the exchange involution, and M, a suitable 4-dimensional subalgebra of the algebra of 4×4 upper triangular matrices.

Hence $\mathrm{var}^*(F \oplus F)$ and $\mathrm{var}^*(M)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety is polynomially bounded.

From their description it follows that there exists no variety with intermediate growth of the $*$-codimensions between polynomial and exponential, i.e, either $c_n^*(\mathcal{V})$ is polynomially bounded or $c_n^*(\mathcal{V})$ grows exponentially. The above 2 algebras play the role of the infinite-dimensional Grassmann algebra and the algebra of 2×2 upper triangular matrices in the ordinary case ([12,13]).

Recently, much interest was put into the study of varieties of polynomial growth (see for instance [3–6, 15,16,14,18]) and different characterizations were given.

In this paper we completely classify all subvarieties of the varieties of $*$-algebras of almost polynomial growth by giving a complete list of finite dimensional $*$-algebras generating them.

Moreover we classify all their minimal subvarieties of polynomial growth, i.e., varieties $\mathcal{V}$ satisfying the property: $c_n^*(\mathcal{V}) \approx qn^k$ for some $k \geq 1, q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$, $c_n^*(\mathcal{U}) \approx q'n^t$ with $t < k$.

2. On star-algebras with polynomial codimension growth

Throughout this paper F will denote a field of characteristic zero and A an associative F-algebra with involution $*$. Let us write $A = A^+ \oplus A^-$, where $A^+ = \{a \in A | \ a^* = a\}$ and $A^- = \{a \in A | \ a^* = -a\}$ denote the sets of symmetric and skew elements of A, respectively. Let $X = \{x_1, x_2, \ldots\}$ be a countable set and let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ be the free associative algebra with involution on X over F. It is useful to regard to $F\langle X, *\rangle$ as generated by symmetric and skew variables: if for $i = 1, 2, \ldots$, we let $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$, then $F\langle X, *\rangle = F\langle y_1, z_1, y_2, z_2, \ldots\rangle$. Recall that a polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle X, *\rangle$ is a $*$-polynomial identity of A (or simply a $*$-identity), and we write $f \equiv 0$, if $f(s_1, \ldots, s_n, k_1, \ldots, k_m) = 0$ for all $s_1, \ldots, s_n \in A^+$, $k_1, \ldots, k_m \in A^-$.

We denote by $\mathrm{Id}^*(A) = \{f \in F\langle X, *\rangle | f \equiv 0 \text{ on } A\}$ the T^*-ideal of $*$-identities of A, i.e., $\mathrm{Id}^*(A)$ is an ideal of $F\langle X, *\rangle$ invariant under all endomorphisms of the free algebra commuting with the involution $*$.

It is well known that in characteristic zero, every $*$-identity is equivalent to a system of multilinear $*$-identities. We denote by

$$P_n^* = \mathrm{span}_F \{ w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i = y_i \text{ or } w_i = z_i, \quad i = 1, \ldots, n \}$$

the vector space of multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$. Hence for every $i = 1, \ldots, n$ either y_i or z_i appears in every monomial of P_n^* at degree 1 (but not both).

The study of $\mathrm{Id}^*(A)$ is equivalent to the study of $P_n^* \cap \mathrm{Id}^*(A)$ for all $n \geq 1$ and we denote by

$$c_n^*(A) = \dim_F \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}, \quad n \geq 1,$$

the n-th $*$-codimension of A.

If A is an algebra with 1, by [2] $\mathrm{Id}^*(A)$ is completely determined by its multilinear proper polynomials. Recall that $f(y_1, z_1, \ldots, y_n, z_n) \in P_n^*$ is a proper polynomial if it is a linear combination of elements of the type