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Generalized robust toric ideals

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ABSTRACT

An ideal I is robust if its universal Gröbner basis is a minimal generating set for this ideal. In this paper, we generalize the meaning of robust ideals. An ideal is defined as generalized robust if its universal Gröbner basis is equal to its universal Markov basis. This article consists of two parts. In the first one, we study the generalized robustness on toric ideals of a graph G. We prove that a toric graph ideal is generalized robust if and only if its universal Markov basis is equal to the Graver basis of the ideal. Furthermore, we give a graph theoretical characterization of generalized robust graph ideals, which is based on terms of graph theoretical properties of the circuits of the graph G. In the second part, we go on to describe the general case of toric ideals, in which we prove that a robust toric ideal has a unique minimal system of generators, or in other words, all of its minimal generators are indispensable.

1. Introduction

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a vector configuration in $\mathbb{Q}^n$ and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup, where $\mathbb{N}A$ is pointed, that is, if $x \in \mathbb{N}A$ and $-x \in \mathbb{N}A$ then $x = \mathbf{0}$. We grade the polynomial ring $\mathbb{K}[x_1, \dots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \dots, m$. For $\mathbf{u} = (u_1, \dots, u_m) \in \mathbb{N}^m$, we define the A-degree of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

The toric ideal I_A associated with A is the prime ideal generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$, see [21].

Toric ideals constitute a special class of ideals in a polynomial ring. They define toric varieties, a large class of algebraic varieties, that play an important role in the development of mathematics the last years. Their study starts with Hochster in [14] and spreads through a series of lectures by Fulton, see [11,12].

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As far as the applicability of toric ideals is concerned, it has to be mentioned that toric ideals are related to recent advances in polyhedral geometry, toric geometry, algebraic geometry, algebraic statistic, integer programming, graph theory, computational algebra, etc., where they are applied in a natural way, see for example [9,10,17,21].

There are several sets for a toric ideal, which include crucial information about it, such as the Graver basis, the universal Markov basis, the universal Gröbner basis and the set of the circuits. An irreducible binomial $\mathbf{x^u} - \mathbf{x^v}$ in I_A is called primitive if there is no other binomial $\mathbf{x^w} - \mathbf{x^z}$ in I_A, such that $\mathbf{x^w}$ divides $\mathbf{x^u}$ and $\mathbf{x^z}$ divides $\mathbf{x^v}$. The set of the primitive binomials forms the Graver basis of I_A and is denoted by Gr_A. As it is known by a theorem of Diaconis and Sturmfels, every minimal generating set of I_A corresponds to a minimal Markov basis of A, which is denoted by M_A, see [9, Theorem 3.1]. The universal Markov basis of A is denoted by $\mathcal{M}_A$ and is defined as the union of all minimal Markov bases of A, see [15, Definition 3.1]. The universal Gröbner basis of an ideal I_A, which is denoted by $\mathcal{U}_A$, is a finite subset of I_A and it is a Gröbner basis for the ideal with respect to all admissible term orders, see [21]. The support of a monomial $\mathbf{x^u}$ of $\mathbb{K}[x_1, \dots, x_m]$ is $\operatorname{supp}(\mathbf{x^u}) := \{i \mid x_i \text{ divides } \mathbf{x^u}\}$ and the support of a binomial $B = \mathbf{x^u} - \mathbf{x^v}$ is $\operatorname{supp}(B) := \operatorname{supp}(\mathbf{x^u}) \cup \operatorname{supp}(\mathbf{x^v})$. An irreducible binomial is called circuit if it has minimal support. The set of the circuits of a toric ideal I_A is denoted by $\mathcal{C}_A$. The relation between the above sets was studied by B. Sturmfels in [21]:

Proposition 1.1. *(See [21, Proposition 4.11].) For any toric ideal I_A it holds:*

$$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A.$$

An ideal I is called robust if its universal Gröbner basis is equal to a Markov basis of the ideal. Robustness is a property of ideals that has not been fully described. More specifically, it has been described for toric ideals which are generated by quadratics. Toric ideals which are generated by quadratics were studied by Ohsugi and Hibi in [18], while the robustness for this class of ideals is described in the article of Boocher and Robeva, see [2]. The importance of robustness stems from the interest in the study of ideals which are minimally generated by a Gröbner basis for an arbitrary term order, see [8]. Moreover, the study of robustness is important, due to the fact that several areas of mathematics are keen on the research of the Markov basis, the universal Gröbner basis and the Graver basis of an ideal. This problem has also been researched in the case of toric ideals arising from a graph G, as studied by Boocher et al. in [3]. In their work the authors proved that any robust toric ideal of a graph G is also minimally generated by its Graver basis, [3, Theorem 3.2]. In addition, they completely characterize all graphs which give rise to robust ideals, see [3, Theorem 4.8].

The present article generalizes the meaning of robust ideals. A robust ideal is called generalized robust if its universal Gröbner basis is equal to its universal Markov basis. This manuscript is divided into two parts.

In the first part, we study the generalized robustness on toric ideals of a graph G. The results of this part are inspired and guided by the work of [3] in order to give theorems that fully characterize the generalized robust toric ideals of graphs. The papers [20,22,24] describe the Markov basis, the Graver basis, the universal Gröbner basis and the set of the circuits for a toric ideal arising from a graph. In Section 2, we analyze all these notions more explicitly. Applying this knowledge on the work of Boocher et al. (see [3]), we are allowed to provide the study of the generalized robustness of graphs, with theorems of the same structure as theirs. In Section 3, we first prove that a toric graph ideal is generalized robust if and only if its universal Markov basis is equal to the Graver basis of the ideal, see Theorem 3.4. Moreover, the relation between robust graph ideals and generalized robust graph ideals is studied. In the next section, we go on to give a graph theoretical characterization of generalized robust graph ideals, which is based on terms of graph theoretical properties of the circuits of the graph G, see Theorem 4.5.