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Zip property of certain ring extensions

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ABSTRACT

The aim of the paper is to investigate the behavior of the right zip property under some ring constructions. It includes actions of Hopf algebras, rings of quotients and subrings of matrix rings.

1. Introduction and preliminary remarks

Throughout the paper a ring means an associative ring with unity. For any subset S of a ring R, $\mathrm{r.ann}_R(S)$ stands for the right annihilator of S in R, i.e. $\mathrm{r.ann}_R(S) = \{r \in R \mid Sr = 0\}$. The left annihilator is denoted by $\mathrm{l.ann}_R(S)$.

A ring R is said to be right zip if, for any subset S of R with $\mathrm{r.ann}_R(S) = 0$, there exists a finite subset S_0 of S such that $\mathrm{r.ann}_R(S_0) = 0$. In the above definition, one can equivalently require that S is a left ideal of R. We will often use this characterization.

The notion of zip rings was introduced by C. Faith in [4] as a specialization of considerations of Zelmanowitz, J.A. Beachy and W.D. Blair (cf. [12,1]). This paper marked the beginning of the systematic studies of, and posing problems on the behavior of the zip property under various algebraic constructions. The subject has been studied by many authors (see for example [3,6,7,13]). The aim of this paper is to investigate the behavior of the zip property under various kinds of ring extensions.

Section 2 is focused on the zip property in the presence of a Hopf algebra action. It is known that there are deep relations between various algebraic properties of an algebra A and its subalgebra of constants A^H under the action of a finite dimensional Hopf algebra H. The relations are especially strong when the smash product $A\#H$ is semiprime. We show that a similar situation concerns the right zip property. Namely A^H

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is right zip if and only if A is right H-zip. We also provide examples showing the limits of the obtained results. In particular, we show that zip and H-zip properties of A are not equivalent. This situation differs from other finiteness conditions such as ACC or DCC on right ideals and their counterparts for H-stable right ideals. As a side effect we obtain an example of a semiprime ring A acted by a finite group G such that A has ACC on right annihilators of G-stable sets but it does not have ACC on right annihilators.

We will work in the much more general context of an action on a ring A into its subring R, by a set of $(R-R)$-bimodule maps. This action is defined at the beginning of Section 2. The reason to work in such generality is twofold. On the one hand, in this way it is clear which properties are responsible for well behavior of the zip property, and on the other the general result offers a wider class of applications.

In Section 3 we investigate the behavior of the zip property with respect to localization and matrix ring construction. F. Cedó (cf. [3]) proved that the zip property lifts neither to the polynomial ring $R[x]$ nor to the matrix ring $M_n(R)$, for $n \geq 2$. However, in both of these cases, it lifts with the additional assumption that R is commutative (cf. [1,3]). The authors of [7] proved similar results for the ring $UT_n(R)$ of all upper triangular matrices over R. In this section we show that, in fact, the ring $M_n(R)$ is right (left) zip if and only if the ring $UT_n(R)$ is such.

We also investigate the zip property of some other subrings of $M_n(R)$. In particular, we show that the ring R is right (left) zip if and only if the ring $DT_n(R)$ consisting of all upper triangular matrices (m_{ij}) such that $m_{ii} = m_{jj}$, for $1 \leq i, j \leq n$, is such.

Concerning localization, we prove that, if $\mathcal{S}$ is a two-sided Ore set of regular elements of a ring R, then R is right (left) zip if and only if $R\mathcal{S}^{-1}$ is such. Examples showing that such equivalence does not hold, when $\mathcal{S}$ is an Ore set only on one side, are provided.

In the following two propositions we collect, without proofs, a few basic properties of right zip rings (see [4] and [3]). Recall that a ring R is left Kasch if every simple left R-module can be embedded into ${}_RR$.

Proposition 1.1.

(a) *Any finite ring is right (and left) zip;*
(b) *A product of two rings $R \times T$ is right zip if and only if R and T are right zip;*
(c) *Any ring satisfying DCC on right annihilators is right zip;*
(d) *Any left noetherian ring is right zip;*
(e) *Any left Kasch ring is right zip.*

Proposition 1.2. *Let R be a subring of a right zip ring T. Then R is right zip in any of the following cases:*

(a) *Every nonzero right ideal of T intersects R non-trivially, i.e. R_R is an essential submodule of the right R-module T_R;*
(b) *T satisfies the descending chain condition on right annihilators;*
(c) *T is free as a left R-module.*

2. Zip-property under G-action

Having in mind relations between the zip property of a ring and its subring of constants under the action of groups or more generally Hopf algebras, we begin this section with a general notion of an action which is suitable for our purposes.

Henceforth $R \subseteq A$ will denote an extension of rings. We say that a subset $G \subseteq \mathrm{End}({}_RA_R)$ of (R, R)-bimodule endomorphisms of A acts on A into R if, for any $\sigma \in G$, $\sigma(1) \in R$.

A subset S of A is called G-stable if $\sigma(S) \subseteq S$, for any $\sigma \in G$. Notice that, due to the assumption that $\sigma(1) \in R$ for $\sigma \in G$, every one-sided ideal of R is G-stable. Examples of such actions will be presented later