Journal of Pure and Applied Algebra 220 (2016) 364–381

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Cycle-finite algebras having finitely many indecomposable modules lying on short paths with injective source and projective target

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ARTICLE INFO

Article history:
Received 9 October 2014
Received in revised form 20 April 2015
Available online 21 July 2015
Communicated by S. Koenig

MSC:
16E10; 16G10; 16G60; 16G70

ABSTRACT

In this article we describe the structure of artin algebras A for which any cycle of indecomposable finitely generated (right) A-modules is finite and almost all indecomposable (finitely generated) A-modules do not lie on short paths with injective source and projective target.

1. Introduction and related background

Throughout the paper, by an algebra we mean a basic and indecomposable artin K-algebra over a fixed commutative artin ring K. Given an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated (right) A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ consisting of all indecomposable modules, and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, E)$, where E is a minimal injective cogenerator in $\operatorname{mod} K$. A significant combinatorial and homological invariant of the module category of an algebra A is its Auslander–Reiten quiver, denoted by Γ_A. Moreover, τ_A and τ_A^{-1} denote, respectively, the Auslander–Reiten operators $D\operatorname{Tr}$ and $\operatorname{Tr} D$. Further, the Jacobson radical of $\operatorname{mod} A$, that is, the ideal of $\operatorname{mod} A$ generated by all nonisomorphisms between modules in $\operatorname{ind} A$, is denoted by rad_A, and the infinite Jacobson radical $\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \in \mathbb{N}$, of rad_A.

Prominent role in the representation theory of algebras is played by paths of modules. Recall that, following Ringel [37], a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n \qquad (*)$$

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http://dx.doi.org/10.1016/j.jpaa.2015.06.017

of homomorphisms in $\operatorname{mod} A$ is said to be a path in $\operatorname{ind} A$, provided that all $f_1, \dots, f_n$ are nonzero nonisomorphisms between indecomposable modules. The path is called a *short path*, if $n = 2$ [36]. We shall also call the module X_0 (respectively, X_n) the source (respectively, the target) of this path. Moreover, by a *cycle* in $\operatorname{ind} A$ we mean a path whose source and target coincide. A cycle $(*)$ is said to be *finite*, if all homomorphisms $f_1, \dots, f_n$ do not belong to $\operatorname{rad}_A^\infty$. Recall that Assem and Skowroński [5] defined an algebra A to be a *cycle-finite* algebra if all cycles in $\operatorname{ind} A$ are finite. The cycle-finite algebras form a large and extensively studied class of algebras containing the following particular classes: algebras of finite representation type, tame tilted algebras [19,37], tubular algebras [37,38], tame quasitilted algebras [21,48], tame double tilted algebras [34], tame generalized double tilted algebras [35], tame coil and multicoil algebras [5,6], tame generalized multicoil algebras [30], and strongly simply connected algebras of polynomial growth [46]. Note also that, we can frequently reduce, using covering techniques, the representation theory of an algebra to the representation theory of cycle-finite algebras which are convex subcategories of some cycle-finite locally bounded K-category (see [32,47], for some general results). Let us only mention that this is the case for selfinjective algebras of polynomial growth over an algebraically closed field (see [50]). We refer to the papers [4,10,26–28,45,52,53], for more results concerning cycle-finite algebras.

In this article we investigate the class of algebras A for which there are at most finitely many modules X in $\operatorname{ind} A$ (up to isomorphism) which lie on short paths in $\operatorname{ind} A$ with injective source and projective target, or equivalently, for all but finitely many isomorphism classes of modules X in $\operatorname{ind} A$, we have $\operatorname{Hom}_A(D(A), X) = 0$ or $\operatorname{Hom}_A(X, A) = 0$. We mention that the short paths of indecomposable modules appeared naturally in the representation theory (see [7,13,15–18,25,36,53]). We discuss briefly the origins and motivations for our study.

The beginnings of the problem reach middle of 90s, when Happel, Reiten and Smalø had introduced the concept of a quasitilted algebra in their paper [14]. Following [14], A is a *quasitilted algebra*, if A is of the form $\operatorname{End}_{\mathcal{H}}(T)$, where T is a tilting object in a hereditary abelian Ext-finite K-category $\mathcal{H}$. We mention that (see [14, Theorem 2.3]) an algebra A is quasitilted if and only if the global dimension $\operatorname{gl.dim} A$ of A is at most two and, for all modules X in $\operatorname{ind} A$, we have $\operatorname{pd}_A X \leqslant 1$ or $\operatorname{id}_A X \leqslant 1$. We shall recall now the following essential observation made in [14], which characterizes quasitilted algebras in terms of paths with injective source and projective target. Namely, it has been proved [14, Theorem 1.14] that A is a quasitilted algebra if and only if every path in $\operatorname{ind} A$ with injective source and projective target has a refinement to a path of irreducible homomorphisms in $\operatorname{ind} A$ and any such refinement is sectional. The result was soon extended by Coelho and Lanzilotta [11], and about three years later, by Skowroński [49].

Recall also that Coelho and Lanzilotta were studying the class of, so called, *algebras with small homological dimensions* (or briefly, *shod* algebras), for which every module X in $\operatorname{ind} A$ satisfies $\operatorname{pd}_A X \leqslant 1$ or $\operatorname{id}_A X \leqslant 1$. It follows clearly from the above mentioned homological characterization of quasitilted algebras that every quasitilted algebra is a shod algebra. Furthermore, it was shown in [11] that the class of shod algebras can be characterized via properties of paths with injective source and projective target, in a similar way as it was done for quasitilted algebras before. Namely, it was proved (see [11, (2.1)]) that the shod algebras are exactly those algebras A for which every path in $\operatorname{ind} A$ with injective source and projective target has a refinement to a path of irreducible homomorphisms in $\operatorname{ind} A$ and any such refinement has at most two hooks (if there are two, then they are consecutive). We refer also to [34] for more results on shod algebras (let us only mention that the class of shod algebras which are not quasitilted coincides with the class of algebras called *double tilted algebras* [34, Section 7]). Note that the concept of a double tilted algebra was generalized by Reiten and Skowroński in [35], where a wide class of algebras called *generalized double tilted algebras* has been introduced and investigated; let us only mention that this class of algebras was also independently studied by Assem and Coelho in [1].

The above results strongly suggest that, for an algebra A, the properties of paths in $\operatorname{ind} A$ with injective source and projective target might have an important impact on the structure of the algebra A, and consequently, on the structure of its module category $\operatorname{mod} A$ and behaviour of related invariants. This kind